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Locally Transferable Congruences*

By

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Abstract

An algebra $\mathfrak{A} = (A, F)$ with a constant 0 has locally transferable congruences if for every $a, b \in A$ there exists an element $c \in A$ such that $\Theta(0, a) = \Theta(b, c)$. We characterize varieties of algebras with locally transferable congruences by Maltsev type conditions. We show that such varieties are always locally regular and that a more general condition than local transferability, the so-called local n-transferability, is equivalent to local regularity.

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The concept of transferable congruences was introduced by the first author in [1]. Recall that an algebra $\mathfrak{A} = (A, F)$ has *transferable congruences* if for every a, b, c of A there exists an element $d \in A$ such that

$$\Theta(a, b) = \Theta(c, d)$$

where $\Theta(x, y)$ denotes the principal congruence generated by the pair (x, y). The paper [1] describes connections of this property with the so-called regularity of congruences. However, regularity was general-

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ized to local regularity (see [2]), and some important algebras and varieties which are not regular appear to be locally regular. Hence, there is a natural question if also the concept of transferability of congruences can be "localized" to be related with local regularity of congruences.

In the whole paper we suppose that every considered algebra or variety has a constant denoted by 0 (i.e. it is an n-ary term operation with a constant value or a nullary operation). At first we recall that the algebra $\mathfrak{A}$ is *locally regular* if for every $\Theta, \Phi \in \mathrm{Con}\,\mathfrak{A}$ we have

$$\text{if}\quad [a]\Theta = [a]\Phi \quad \text{for some}\quad a \in A \quad \text{then}\quad [0]\Theta = [0]\Phi.$$

In other words, the 0-class of every congruence is determined by any other congruence class. A variety $\mathcal{V}$ is *locally regular* if each $\mathfrak{A} \in \mathcal{V}$ has this property. The following result was proved in [2] (and is also contained in [4]):

Proposition 1. *A variety $\mathcal{V}$ is locally regular if and only if there exist an integer $n \geq 1$ and binary terms $t_1(x,y), \ldots, t_n(x,y)$ such that*

$$t_1(x,y) = \cdots = t_n(x,y) = y \quad \textit{if and only if} \quad x = 0$$

holds in every algebra of $\mathcal{V}$.

Although this characterization is rather simple, we still do not know in general how the structure of the variety $\mathcal{V}$ influences the number n of binary terms in question. However, we will show that the case $n = 1$ is equivalent to the property stated in the following

Definition 1. An algebra $\mathfrak{A}$ has *locally transferable congruences* if for every $a, b \in A$ there exists an element $c \in A$ such that $\Theta(0,a) = \Theta(b,c)$. A variety $\mathcal{V}$ has *locally transferable congruences* if each $\mathfrak{A} \in \mathcal{V}$ has this property.

We characterize varieties having locally transferable congruences as follows:

Theorem 1. *For a variety $\mathcal{V}$, the following conditions are equivalent*:

(1) *$\mathcal{V}$ has locally transferable congruences.*

(2) *There exist an integer $k \geq 1$, a binary term t and ternary terms $p_1, \ldots, p_k$ such that*

$$\begin{aligned} t(0,y) &= y, \\ x &= p_1(x,y,y), \\ p_i(x,y,t(x,y)) &= p_{i+1}(x,y,y) \quad \textit{for} \quad i = 1, \ldots, k-1 \end{aligned}$$

and

$$0 = p_k(x, y, t(x, y)).$$

(3) *There exists a binary term t such that*

$$t(x, y) = y \quad \text{if and only if} \quad x = 0$$

holds in every algebra of $\mathcal{V}$.

Proof. (1) ⇒ (2): Suppose that $\mathcal{V}$ has locally transferable congruences and let $\mathfrak{A} = F_{\mathcal{V}}(x, y)$ be the free algebra of $\mathcal{V}$ generated by $\{x, y\}$. Then there exists a binary term $t = t(x, y) \in A$ such that $\Theta(x, 0) = \Theta(y, t)$. Hence, $(y, t(x, y)) \in \Theta(x, 0)$ which yields immediately $t(0, y) = y$. Moreover, $(x, 0) \in \Theta(y, t(x, y))$ and applying the Maltsev lemma for the description of the principal congruence $\Theta(y, t(x, y))$ of $\mathfrak{A}$ (see e.g. [4]) we obtain ternary terms $p_1, \ldots, p_k$ such that the desired identities hold in $\mathcal{V}$.

(2) ⇒ (3): Consider the term $t(x, y)$ of (2). Then $t(0, y) = y$. Conversely, suppose $t(x, y) = y$. Then, applying the identities of (2), we have

$$x = p_1(x, y, y) = p_1(x, y, t(x, y)) = p_2(x, y, y) = \cdots = 0$$

proving (3).

(3) ⇒ (1): Let $\mathfrak{A} = (A, F) \in \mathcal{V}$ and $a, b \in A$. Set $c = t(a, b)$ where $t(x, y)$ is the term of (3). Then

$$(b, c) = (t(0, b), t(a, b)) \in \Theta(0, a).$$

Let $\Theta = \Theta(b, c)$ and consider the quotient algebra $\mathfrak{A}/\Theta$. Then $\mathfrak{A}/\Theta \in \mathcal{V}$ and hence

$$[b]\Theta = [c]\Theta = [t(a, b)]\Theta = t([a]\Theta, [b]\Theta)$$

in $\mathfrak{A}/\Theta$ thus, by (3), $[a]\Theta = [0]\Theta$ giving $(a, 0) \in \Theta(b, c)$. Altogether, we have shown $\Theta(0, a) = \Theta(b, c)$ proving (1). □

Corollary 1. *If a variety* $\mathcal{V}$ *has locally transferable congruences, then* $\mathcal{V}$ *is locally regular.*

The proof follows immediately by comparing (3) of Theorem 1 with Proposition 1 (for $n = 1$). □

Example 1. An *ortholattice* is a bounded lattice with a unary operation $^\perp$ satisfying the identities: $(x^\perp)^\perp = x$, $x \wedge x^\perp = 0$, $x \vee x^\perp = 1$, $(x \vee y)^\perp = x^\perp \wedge y^\perp$ and $(x \wedge y)^\perp = x^\perp \vee y^\perp$. The variety of all ortholattices has locally transferable congruences. To verify this, we observe that the term

$$t(x,y) = (x \wedge y^{\perp}) \vee (x^{\perp} \wedge y)$$

satisfies condition (3) of Theorem 1:

$$t(0,y) = (0 \wedge y^{\perp}) \vee (0^{\perp} \wedge y) = 0 \vee (1 \wedge y) = y.$$

Conversely, suppose $t(x,y) = y$, i.e.

$$(x \wedge y^{\perp}) \vee (x^{\perp} \wedge y) = y. \tag{i}$$

Then clearly $x \wedge y^{\perp} \leq y$ and hence

$$x \wedge y^{\perp} = (x \wedge y^{\perp}) \wedge y^{\perp} \leq y \wedge y^{\perp} = 0$$

giving

$$x \wedge y^{\perp} = 0. \tag{ii}$$

By substituting (ii) into (i) we obtain $x^{\perp} \wedge y = y$ whence $x^{\perp} \geq y$ which implies $x \leq y^{\perp}$. Together with (ii), this yields

$$x = x \wedge y^{\perp} = 0.$$

Let us note that the term t is not unique. Similarly as above, one can easily check that also the term

$$t_0(x,y) = (x \vee y) \wedge (x^{\perp} \vee y^{\perp})$$

satisfies $t_0(x,y) = y$ if and only if $x = 0$.

Recall that an algebra $\mathfrak{A}$ is *permutable at* 0 if $[0](\Theta \cdot \Phi) = [0](\Phi \cdot \Theta)$ for every $\Theta, \Phi \in \mathrm{Con}\ \mathfrak{A}$. A variety $\mathcal{V}$ is *permutable at* 0 if each $\mathfrak{A} \in \mathcal{V}$ has this property.

The following characterization was involved in [6] (and is also contained in [4]).

Proposition 2. *A variety $\mathcal{V}$ is permutable at* 0 *if and only if there exists a binary term $s(x,y)$ such that the following identities hold in $\mathcal{V}$:*

$$s(x,x) = 0 \qquad \textit{and} \qquad s(x,0) = x.$$

Example 2. The variety of ortholattices is permutable at 0. To verify this, we can check that $s(x,y) = x \wedge y^{\perp}$ satisfies the identities of Proposition 2.

Similarly, the variety of pseudo-complemented semilattices is permutable at 0 which is witnessed by $s(x,y) = x \wedge y^{*}$.

We are going to show how permutability at 0 can simplify the condition (2) of Theorem 1.

Theorem 2. *For a variety $\mathcal{V}$, the following conditions are equivalent*:

(1) *$\mathcal{V}$ is permutable at* 0 *and has locally transferable congruences.*

(2) *There exist a binary term t and a ternary term p such that*

$$t(0,y) = y, \qquad x = p(x,y,y) \qquad \textit{and} \qquad 0 = p(x,y,t(x,y)).$$

Proof. (1) ⇒ (2): Consider $\mathfrak{A} = F_{\mathcal{V}}(x,y)$, the free algebra of $\mathcal{V}$ generated by $\{x,y\}$. As in the proof of Theorem 1 there exists a binary term $t = t(x,y) \in A$ such that $\Theta(x,0) = \Theta(y,t)$ and thus $t(0,y) = y$ and $(x,0) \in \Theta(y,t(x,y))$, i.e. $x \in [0]\Theta(y,t(x,y))$. Since $\mathcal{V}$ is permutable at 0, we can apply Lemma 2.1 of [5] (which is also contained in [4]) to obtain $x \in [0]R(y,t(x,y))$, where $R(a,b)$ denotes the least reflexive and compatible binary relation on $\mathfrak{A}$ containing the pair (a,b). Hence, $(x,0) \in R(y,t(x,y))$, thus there is a unary polynomial φ over $\mathfrak{A}$ such that $x = \varphi(y)$ and $0 = \varphi(t(x,y))$ (see e.g. [4]). Since $\mathfrak{A}$ is generated by $\{x,y\}$, there exists a ternary term p such that $\varphi(z) = p(x,y,z)$ thus $x = p(x,y,y)$ and $0 = p(x,y,t(x,y))$.

(2) ⇒ (1): Applying (2) of Theorem 1 for $k = 1$ we conclude that $\mathcal{V}$ has locally transferable congruences. Further, set

$$s(x,y) = p(x,x,t(y,x)).$$

Then $s(x,x) = p(x,x,t(x,x)) = 0$, $s(x,0) = p(x,x,t(0,x)) = p(x,x,x) = x$ by (2) and, in account of Proposition 2, $\mathcal{V}$ is permutable at 0. □

We introduce a concept which is more general than local transferability of congruences:

Definition 2. An algebra $\mathfrak{A}$ has *locally n-transferable congruences* ($n \geq 1$) if for every $x,y \in A$ there exist $c_1,\dots,c_n \in A$ such that $\Theta(0,x) = \Theta(y,c_1,\dots,c_n)$. A variety $\mathcal{V}$ has *locally n-transferable congruences* if each $\mathfrak{A} \in \mathcal{V}$ has this property.

Let us mention that $\Theta(y,c_1,\dots,c_n)$ is the least congruence containing the set $\{y,c_1,\dots,c_n\} \times \{y,c_1,\dots,c_n\}$ and hence $\Theta(y,c_1,\dots,c_n) = \Theta(y,c_1) \vee \dots \vee \Theta(y,c_n)$.

We show that by using local n-transferability instead of local transferability the assertion of our Corollary 1 can be converted:

Theorem 3. *For a variety $\mathcal{V}$, the following conditions are equivalent*:

(1) *There exists an integer $n \geq 1$ such that $\mathcal{V}$ has locally n-transferable congruences.*

(2) *$\mathcal{V}$ is locally regular.*

Proof. (1) ⇒ (2): Let $\mathcal{V}$ have locally n-transferable congruences for some $n \geq 1$. Consider the free algebra $F_{\mathcal{V}}(x,y)$ generated by $\{x,y\}$.

Then there exist $c_1, \ldots, c_n \in F_{\mathcal{V}}(x,y)$ such that $\Theta(0,x) = \Theta(y, c_1, \ldots, c_n)$. Of course, $c_i = t_i(x,y)$ $(i = 1, \ldots, n)$ for some binary terms $t_1, \ldots, t_n$. Hence

$$\Theta(0,x) = \Theta(y, t_1(x,y)) \vee \cdots \vee \Theta(y, t_n(x,y)).$$

Therefore, $y = t_i(0,y)$ for $i = 1, \ldots, n$ and, conversely, if $y = t_i(x,y)$ for all indices i then $x = 0$. By Proposition 1, $\mathcal{V}$ is locally regular.

(2) $\Rightarrow$ (1): Let $\mathcal{V}$ be locally regular and consider again $\mathfrak{A} = F_{\mathcal{V}}(x,y)$. Put $C = [y]\Theta(0,x)$ and $\Phi = \Theta(\{y\} \times C)$. Then Φ and $\Theta(0,x)$ have the class C in common and, due to local regularity, also $[0]\Phi = [0]\Theta(0,x)$. Hence,

$$(0,x) \in \Theta(\{y\} \times C).$$

By a standard application of the Maltsev lemma describing the congruence $\Theta(\{y\} \times C)$ (see [4] and [3], Theorem 3.2) we infer the existence of binary terms $t_1, \ldots, t_n \in C$ and 4-ary terms $q_1, \ldots, q_n$ $(n \geq 1)$ such that

$$\begin{aligned} t_i(0,y) &= y && \text{for} \quad i = 1, \ldots, n, \\ 0 &= q_1(y, t_1(x,y), x, y), \\ q_j(t_j(x,y), y, x, y) &= q_{j+1}(y, t_{j+1}(x,y), x, y) && \text{for} \quad j = 1, \ldots, n-1 \end{aligned}$$

and

$$x = q_n(y, t_n(x,y), x, y).$$

Now, let $\mathfrak{A} = (A, F) \in \mathcal{V}$, $a, b \in A$ and $c_i = t_i(a,b)$ $(i = 1, \ldots, n)$. Clearly

$$(b, c_i) = (t_i(0,b), t_i(a,b)) \in \Theta(0,a),$$

thus

$$\Psi = \Theta(b, c_1, \ldots, c_n) \subseteq \Theta(0,a).$$

Conversely,

$$\begin{aligned} 0 &= q_1(b, t_1(a,b), a, b) \Psi q_1(c_1, b, a, b) \\ &= q_2(b, c_2, a, b) \Psi q_2(c_2, b, a, b) \\ &= q_3(b, c_3, a, b) \Psi \cdots = a \end{aligned}$$

giving $(0,a) \in \Psi = \Theta(b, c_1, \ldots, c_n)$. Together, $\Theta(0,a) = \Theta(b, c_1, \ldots, c_n)$ thus $\mathfrak{A}$ and hence $\mathcal{V}$ have locally n-transferable congruences. $\square$

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