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The Evolution of Fluctuations in the Laser Model

By

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Abstract

The time evolution of a system consisting of two-level atoms and a laser field is compared with the evolution of a mean field theory. The evolution on the quasilocal level is extended to the fluctuation algebra, where with the modifications necessary for time-dependent states it can be described as a quasifree automorphism. The spectral properties of these automorphisms are related to the stability properties of the underlying quasilocal state.

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1. Introduction

The laser model describing the interaction of a laser radiation with matter consisting of two-level atoms is both of physical interest as of mathematical subtlety. It was investigated starting with [1–3] and finally treated with all mathematical rigor in [4] showing interesting features like phase transition. We are here interested in its simplest version, i.e. we take into account only one laser mode and ignore all interactions except the interplay of the laser field with the atoms. Already here a nontrivial time evolution results. For finitely many atoms the time evolution is given by a Hamiltonian and therefore corresponds to an automorphism of the underlying von Neumann algebra. In the thermodynamic limit, when the number of atoms tends to infinity, there exist several limits. We can consider the expectation value of a single spin and that of a finite product of spins. Then the relevant algebra is the quasilocal $\mathcal{C}^*$ algebra influenced by the Weyl algebra of the laser field. For this Weyl algebra we take for the thermodynamic limit a sequence of states on the laser field with photon number increasing proportional to the number of atoms but at the same time with increasing correlations between the photons so that the entropy of the laser field remains finite. For this sequence of states the evolution of the expectation values of characteristic quantities in the course of time was evaluated in [5], also similar results though in a wider context can be found in [6], [7]. We obtain a time evolution on the quasilocal level. The fact that we are dealing with an automorphism group is lost in the limit due to the scaling of the Hamiltonian.

In [4] however the bosonic creation and annihilation operators of the laser field are replaced by bounded operators in an appropriate scaling. The advantage is that some limits can be controlled more easily. We will call this model mean field model. As an additional advantage we will note that the time evolution remains an automorphism on the quasilocal algebra though this automorphism is time-dependent and does not preserve the group structure. In this note we want to restore this automorphism property to some extent also for the laser model. In order to find an automorphism for the whole system we have to include the algebra of fluctuations of the atoms, therefore studying the time evolution on a mesoscopic scale. The algebra of fluctuations was introduced in [8]. Its time evolution was discussed for time-invariant states and interactions with finite range, where it is closely related to the time evolution on the quasilocal level. But already in [9] it was observed that the time evolution shows new features if it results from a mean field theory, even if the state remains invariant in time. This fact was applied in experiment [10] to construct mesoscopic entanglement. This experiment was analyzed in the framework of the fluctuation algebra in [11]. Therefore it is evident that the fluctuation algebra can change though the change cannot be observed locally. Here we will show that provided the state is invariant in time, i.e. our setting corresponds to the setting of [10] the evolution of the laser field can be understood as the time evolution of the fluctuation algebras of two systems interacting by a mean field Hamiltonian, identifying the fluctuations of one system with the laser field. If the state is not invariant then the mean field time evolution describes a time evolution different from the one with a laser mode, mainly because it does not allow that the number of photons changes. But on the quasilocal level the two time evolutions have such a strong similarity that it is natural to extend the time evolution of matter and laser also to the fluctuation algebra of the matter. The fact that on the quasilocal level it is possible to describe the evolution by an automorphism makes it possible to define a natural map between the fluctuation algebras corresponding to different times. But this map does not describe the time evolution of the fluctuation algebra. It has to be combined with the action of the Hamiltonian on the fluctuations. We obtain in this way a quasi-free time evolution on the Weyl algebra of fluctuations, that is not trivially related to the time automorphism on the quasilocal algebra. For the special case that the underlying quasilocal state is invariant or periodic in time, the evolution on the fluctuation level satisfies the group property and therefore can be expressed by an effective Hamiltonian that is quadratic. For invariant states this Hamiltonian is bounded from below if the invariant state is stable, so that the fluctuations though not invariant remain bounded. For an unstable state the fluctuations increase exponentially. If we consider periodic states then they are always stable and again the corresponding Hamiltonian is bounded from below but now admits a zero frequency, that corresponds to a linear increase of the correlations between the laser field and the fluctuations in the course of time and consequently in an increase of the entropy of the state of the laser algebra. We observe therefore that though the time evolutions on the quasilocal level and on the level of the fluctuations are different they show the same stability behavior.

2. The Quasilocal Model

We concentrate on the simplest example treated in [4] and [5], namely a system of N atoms with two energy levels interacting with a one mode laser. To be more precise, we are interested in the effect of a field of many photons with strong correlations on a quasilocal spin system. We describe the time evolution by a Hamiltonian

$$H_N^1(\epsilon) = \epsilon \sum_{j=1}^{N} \sigma_z^j + \lambda \frac{1}{\sqrt{N}} \sum_{j=1}^{N} [(\sigma_x^j + i\sigma_y^j)b^* + (\sigma_x^j - i\sigma_y^j)b] + kb^*b, \tag{1}$$

where $\vec{\sigma}^j$ are Pauli matrices scaled as $[\sigma_x, \sigma_y] = i\sigma_z$ and b, b^* are bosonic annihilation and creation operators satisfying $[b, b^*] = 1$. In the following we will also study a Hamiltonian of a mean field model describing the coupling of two spin systems, the τ^k also being Pauli matrices,

$$H_N^2 = \epsilon \sum_{j=1}^{N} \sigma_z^j + \frac{\lambda}{N} \sum_{j,k=1}^{N} (\sigma_x^j \tau_x^k + \sigma_y^j \tau_y^k). \tag{2}$$

We can consider this Hamiltonian again to describe the interaction of atoms with a laser in so far as the laser consists of photons to whom we can assign as individual property their polarization that we can relate to the Pauli matrices τ. We assume that we start with M photons and that this number is not changed in the course of time. In our Hamiltonian we have restricted ourselves for simplicity to $N = M$, but the calculations would be similar if M increases proportional to N. What we miss in the second model is the possibility that photons are created or annihilated, what we gain is the mathematical facility that τ is a bounded operator and that the scaling in the Hamiltonian is the natural scaling in a mean field theory. We will see in the following that the two models describe a very similar behavior.

We assume that the system is initially in a state that factorizes in the lattice points of the atom system

$$\omega(\vec{\sigma}^k) = \vec{s}, \qquad \omega(\Pi_j \vec{c}^{k_j} \vec{\sigma}^{k_j}) = \Pi_j (\vec{c}^{k_j} \vec{s}), \qquad \vec{c}^k \in R^3. \tag{3}$$

In the mean field model the same holds for the τ field

$$\omega(\vec{\tau}^k) = \vec{a}, \qquad \omega(\Pi_j \vec{c}^{k_j} \vec{\tau}^{k_j}) = \Pi_j (\vec{c}^{k_j} \vec{a}). \tag{4}$$

Finally we assume that there are initially no correlations with the laser field

$$\omega(\vec{\sigma}^k \vec{\tau}^l) = \vec{s}\vec{a}. \tag{5}$$

If the laser field is described with creation and annihilation operators then again

$$\omega_N(\vec{\sigma}^k e^{i\alpha b + \overline{\alpha} b^*}) = \vec{s}\omega_N(e^{i\alpha b + \overline{\alpha} b^*}), \tag{6}$$

where ω_N is a Gaussian state on the laser system. For $N \to \infty$ the mean photon number will increase so that we need the existence of

$$\lim_{N\to\infty} \frac{\omega_N(b)}{\sqrt{N}} = a e^{i\gamma}, \qquad \lim_{N\to\infty} \frac{\omega_N(b^* b)}{N} = a^2. \tag{7}$$

For such a sequence of states the time evolution of the state determined by H_N^1 respectively by H_N^2 converges in the limit $N \to \infty$ in the following sense:

Theorem 1. *Assume that for the laser model the state satisfies* (3), (6), (7) *and that the time evolution is governed by* (1) *with* $\lambda = 1/\sqrt{2}$. *Then*

$$\begin{aligned}
\lim_{N\to\infty} \omega(e^{iH_N^1 t}\vec{\sigma}e^{-iH_N^1 t}) &= \vec{s}(t), \\
\lim_{N\to\infty} \omega(e^{iH_N^1 t}\sigma_{j_k}^k \cdots \sigma_{j_l}^l e^{-iH_N^1 t}) &= s_{j_k}(t)\cdots s_{j_l}(t), \\
\lim_{N\to\infty} \frac{1}{\sqrt{N}}\omega_N(e^{iH_N^1 t}(b^* + b)e^{-iH_N^1 t}) &= \sqrt{2}a_x(t), \\
\lim_{N\to\infty} \frac{1}{\sqrt{N}}\omega_N(e^{iH_N^1 t}\vec{\sigma}^k(b^* + b)e^{-iH_N^1 t}) &= \sqrt{2}\vec{s}(t)a_x(t), \\
\lim_{N\to\infty} \frac{1}{\sqrt{N}}\omega_N(e^{iH_N^1 t}(ib^* - ib)e^{-iH_N^1 t}) &= \sqrt{2}a_y(t), \\
\lim_{N\to\infty} \frac{1}{\sqrt{N}}\omega_N(e^{iH_N^1 t}\vec{\sigma}^k(ib^* - ib)e^{-iH_N^1 t}) &= \sqrt{2}\vec{s}(t)a_y(t),
\end{aligned} \tag{8}$$

where the vectors $\vec{s}(t)$ *and* $\vec{a}(t)$ *have to be determined by the differential equation*

$$\begin{aligned}
&\frac{d}{dt}s_z = -s_y a_x + s_x a_y, \\
&\frac{d}{dt}s_x = \epsilon s_y - s_z a_y, \qquad \frac{d}{dt}a_x = -s_y - k a_y, \\
&\frac{d}{dt}s_y = -\epsilon s_x + s_z a_x, \qquad \frac{d}{dt}a_y = s_x + k a_x.
\end{aligned} \tag{9}$$

Proof. The differential equation (9) has as constant of the motion

$$s_x^2 + s_y^2 + s_z^2 = \text{const}, \qquad a_x^2 + a_y^2 - s_z = \text{const}. \tag{10}$$

According to the expected time evolution the state factorizes in the lattice points. We can describe the state by the same density matrix at every lattice point, $\rho^k(t) = (1 + \vec{s}(t)\vec{\sigma}^k)/2$. Since $|\vec{s}(t)| = s$, we can write $\rho^k(t) = \rho^k \circ \alpha_t^k$ where α_t^k is an automorphism at the lattice point k rotating the state. Therefore for the quasilocal state $\omega_t = \omega \circ \alpha_S^t$. Here α_S^t are automorphisms of the lattice algebras corresponding to the atoms considered as quasilocal $\mathcal{C}^*$ algebras. This automorphism is strictly local and we consider it to be implemented for the local algebra over a subset $[0,N]$ by $U_N(t)$. Notice however, that the $U_N(t)$ do not satisfy the group property, $U_N(t_1+t_2) \neq U_N(t_1)U_N(t_2)$. If we define the 3×3 matrix $V(t)$ by $\vec{s}(t) = V(t)\vec{s}(0)$ we have $\tilde{H}_N(0)U_N(t) = U_N(t)\tilde{H}_N(t)$ where $\tilde{H}_N(0) = \sum_{j=0}^N \vec{\sigma}^j\vec{a}$ and $\tilde{H}_N(t) = \sum_{j=0}^N (V(t)\vec{\sigma}^j)\vec{a}$. If we now estimate

$$\begin{aligned}\omega(Ae^{iH_N^1 t}U_N(t)^* B) &= \omega(AB) + \int_0^t dt' i\langle\Omega|AB' e^{iH_N^1 t'}(H_N^1 - \tilde{H}(0))U_N(t')|\Omega\rangle \\ &= \omega(AB) + \int_0^t dt' i\langle\Omega|AB' e^{iH_N^1 t'}U_N(t')(H_N^1(t') \\ &\quad - \tilde{H}(t'))|\Omega\rangle,\end{aligned}$$

$$H_N^1(t') - \tilde{H}_N(t') = \sum \vec{\sigma}\left(\frac{\vec{b}}{\sqrt{N}} - \vec{a}\right). \tag{11}$$

Here A, B are operators of $\mathcal{A}_{[0,N]}(\sigma)$ and we have used that the state is cyclic and separating (we assume $|\vec{s}| < 1$) so that B can be replaced by an operator B' from the commutant that can be commuted through. Further $\tilde{H}(t) = \sum \vec{\sigma}^j \vec{c}(t)$ implements the rotation of $\vec{\sigma}^j$ at the time t and is therefore determined by (8) and (9) whereas $H_N^1(t)$ is the total Hamiltonian with rotated $\vec{\sigma}$. Finally it remains to proof strong convergence $\lim_{N\to\infty}(H_N^1(t) - \tilde{H}(t))|\Omega\rangle = 0$. Here the t-dependence is fixed on the basis of the differential equation and is therefore under control. We use the fact that

$$st\text{-}\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N \vec{\sigma}^j|\Omega\rangle = \vec{s}|\Omega\rangle$$

together with the clustering properties (6). It follows that $\lim_{N\to\infty}\gamma_b(e^{iH_N^1 t'}U_N(t')) = 1$ where γ_b is the conditional expectation over the b-field with respect to the given state ω and the limit can be taken in the strong topology of the quasilocal σ-field. To obtain the correct time behavior for the b-field we estimate

$$\begin{aligned}&\omega_N\left(e^{iH_n^1 t}e^{-itkb^*b}\left(\frac{b_x}{\sqrt{N}} - a_x(t)\right)e^{itkb^*b}e^{-itH_N^1}\right) \\ &\quad = \int_0^t dt'\omega_N((e^{iH_n^1 t'}e^{-it'kb^*b}(\sigma_y - s_y)e^{itkb^*b}e^{-itH_N^1}).\end{aligned} \tag{12}$$

Here we profit from the fact that in the commutation relation between H_N^1 and b we remain only with σ for which we already control the time evolution. Compare also the related result in a larger context in [6] and [7].

Theorem 2. *Assume that in the mean field model the state satisfies* (3), (4), (5) *and that the time evolution is determined by* (2) *with* $\lambda = 1$. *Then*

$$\lim_{N\to\infty} \omega(e^{iH_N^1 t}\sigma_{j_k}^k \cdots \sigma_{j_l}^l \tau_{j_m} \cdots \tau_{j_n} e^{-iH_N^1 t}) = s_{j_k}(t) \cdots s_{j_l} a_{j_m} \cdots a_{j_n}(t), \qquad (13)$$

where $s_j(t)$ *and* $a_j(t)$ *are solutions of the differential equations*

$$\begin{aligned} \frac{d}{dt}s_z &= -s_y a_x + s_x a_y, & \frac{d}{dt}a_z &= s_y a_x - s_x a_y, \\ \frac{d}{dt}s_x &= \epsilon s_y - s_z a_y, & \frac{d}{dt}a_x &= -a_z s_y, \\ \frac{d}{dt}s_y &= -\epsilon s_x + s_z a_x, & \frac{d}{dt}a_y &= a_z s_x. \end{aligned} \qquad (14)$$

Proof. Here we have as constant of the motion

$$s_z + a_z = \text{const}, \quad s_x^2 + s_y^2 + s_z^2 = \text{const}, \quad a_x^2 + a_y^2 + a_z^2 = \text{const}. \qquad (15)$$

As before the state restricted to the σ-field, now together with the τ-field can be written $\omega_t = \omega \circ \alpha^t$. Again we consider this automorphism to be implemented on the local level by $U_N(t)$. In (11) we can now take A, B to be local operators belonging both to the σ- and the τ-field. Otherwise the argument remains unchanged. We are permitted to arrange an appropriate c-number in $U_N(t)$ and estimate

$$\begin{aligned} &\lim_{N\to\infty} (H_N^2 - \tilde{H}(0))|\Omega\rangle \\ &\quad = \lim_{N\to\infty} \sum \frac{1}{N}(\sigma_x^j \tau_x^k + \sigma_y^j \tau_y^k) - \sum(\sigma_x^j a_x + \sigma_y a_y) \\ &\qquad - \sum(\tau_x^k s_x + \tau_y^k s_y) - (s_x a_x + s_y a_y)N|\omega\rangle = 0 \end{aligned} \qquad (16)$$

so that again we control the time evolution on the quasilocal level.

Let us characterize once more the time evolution in the thermodynamic limit resulting from (2): We obtain a rotation on every lattice point, but the time evolution is not unitarily implementable since weak limit points are state-dependent. The individual α_S^t, α_T^t do not satisfy the group property reflecting the interaction between atoms and photons. Notice that due to the fact that the time evolution of the quasilocal state is the result of automorphisms on the $\mathcal{C}^*$ algebra the mean entropy of the quasilocal state ω_t remains unchanged in the course of time.

If we turn to the time evolution corresponding to H_N^1 then again we have similar constants of the motion (10). This guarantees that the time evolution with respect to the atoms reduces to a time-dependent automorphism. However due to the scaling it is difficult to interpret the time evolution of the laser field as an automorphism. In addition we did not determine how $\omega_t(b^*b)$ behaves in the course of time, which on one hand is necessary to describe the time evolution on the algebraic level and of course also tells us how the number of photons changes in time. In fact we will see that the time evolution can be interpreted as automorphism only if we add the fluctuation algebra of the atoms which we have to discuss next.

3. The Fluctuation Algebra and Its Time Evolution

Following [8] we can consider the limit

$$\lim_{N\to\infty} \omega\left(e^{i\frac{\sum \vec{a}(\vec{\sigma}^k - \omega(\vec{\sigma}^k))}{\sqrt{N}}} e^{i\frac{\sum \vec{b}(\vec{\sigma}^k - \omega(\vec{\sigma}^k))}{\sqrt{N}}} \right) = \omega(e^{i\vec{a}\vec{S}} e^{i\vec{b}\vec{S}}). \tag{17}$$

In [8] it was shown that this defines new operators $W_\omega(\vec{a}\vec{\sigma}) = e^{i\vec{a}\vec{S}}$ that form a Weyl algebra, where the commutation relations are determined by the state, namely

$$e^{i\vec{a}\vec{S}} e^{i\vec{b}\vec{S}} = e^{-\omega([\vec{a}\vec{S},\vec{b}\vec{S}])} e^{i\vec{b}\vec{S}} e^{i\vec{a}\vec{S}}. \tag{18}$$

This Weyl algebra is called fluctuation algebra and is state-dependent. In addition the construction not only defines the operators but at the same time gives a state ω (we use the same letter) on this fluctuation algebra that in fact is a Gaussian state. For the τ algebra we can consider the same procedure and obtain another Weyl algebra $(e^{i\vec{a}\vec{T}})$. If however we start with the laser algebra (b, b^*) and assume together with

$$\lim_{N\to\infty} (\omega_N(b) - \sqrt{N} a e^{i\gamma}) = 0$$

that

$$\lim_{N\to\infty} (\omega_N(b^*b) - a^2 N) < \infty,$$

then we can define new creation and annihilation operators

$$A^* = \lim_{N\to\infty} (b^* - a e^{-i\gamma}\sqrt{N}). \tag{19}$$

Their expectation values are zero for $t = 0$ and

$$\omega(A^*A) = \lim_{N\to\infty} (\omega_N(b^*b) - a^2 N) \tag{20}$$

exists according to our assumption. Evidently (A^*, A) form again a Weyl algebra, that according to the definition we can also call fluctuation algebra of the laser. Together with the fluctuation algebra of the atoms we have in both situations, whether we have fixed the photon number or not, two Weyl algebras, one for the atoms and one for the laser, that are combined to a tensor product and are in a Gaussian state.

If the state of the underlying system evolves in time then the commutation relations of $e^{i\vec{a}\vec{S}}$ change, since their definition is state-dependent. However there is always a spin direction in which the state of the σ field is oriented, $\omega_t(\vec{\sigma}) = \vec{s}(t)$ and in fact in our situation $|\vec{s}(t)| = s$ independent of t (remember the constant of the motion $(s_x^2 + s_y^2 + s_z^2)$). Therefore the Weyl algebras are isomorphic, where the isomorphism rotates $\vec{s}_0$ into $\vec{s}_t$. The corresponding statement is true for $e^{i\vec{b}\vec{T}}$ whereas $e^{i(\alpha A + \overline{\alpha} A^*)}$ determined by (19) does not change in time.

More precisely we generalize (17) and consider the Weyl operator in the fluctuation algebra corresponding to ω

$$W_\omega(C) = w\text{-}\lim e^{i \sum_{j=1}^{N} \frac{C^j - \omega(C^j)}{\sqrt{N}}}$$

in its dependence on C, where C is some quasilocal operator and C^j its translate. The weak limit is taken in the state ω. The Weyl algebra $\mathcal{W}_\omega$ consists of all equivalence classes of $W_\omega(C)$, i.e. we identify $W_\omega(C)$ with $W_\omega(D)$ if $\omega(W_\omega(C) W_\omega(-D)) = 1$. Defining a time evolution we might think of using

$$e^{iH_N t} e^{i \sum \frac{C^j - \omega(C^j)}{\sqrt{N}}} e^{-iH_N t}.$$

But here there appear several difficulties. First the Weyl operator

$$e^{i \sum \frac{C^j - \omega(C^j)}{\sqrt{N}}}$$

converges only in a weak sense. Time evolution that changes the underlying quasilocal state destroys the convergence. Therefore we have to subtract the correct value moving to the appropriate fluctuation algebra. Next we have to take care that the limits $N \to \infty$ in the Hamiltonian and for the fluctuation algebra are coupled and this can have the consequence that the time evolution of the quasilocal algebra and of the fluctuation algebra are not trivially related. Therefore we divide the time evolution into several steps that we hope will preserve the algebraic relations so that in the final estimate we can use unitarity to control convergence.

Definition 1. Consider an automorphism α_t on the underlying quasi-local algebra. This automorphism defines first a map between the states, $\omega_t = \omega \circ \alpha_t$, and secondly an isomorphism $\hat{\alpha}_t$ between the Weyl algebra $\mathcal{W}_\omega$ and $\mathcal{W}_{\omega_t}$ via

$$\hat{\alpha}_t W_\omega(C) = W_{\omega_t}(\alpha_{-t} C). \tag{21}$$

Remark. To show that $\hat{\alpha}_t$ is in fact an isomorphism all we need is to control the Weyl relations

$$\begin{aligned} W_\omega(C) W_\omega(D) W_\omega(C)^{-1} W_\omega(D)^{-1} &= e^{-\omega[C,D]} \\ &= \hat{\alpha}_t (W_\omega(C) W_\omega(D) W_\omega(C)^{-1} W_\omega(D)^{-1}). \end{aligned}$$

which is evidently satisfied.

This does not allow yet to talk about time correlations inside of the fluctuation algebra. These correlations can easily be defined if ω is invariant under α_t.

Definition 2. Let α_t be an automorphism on the underlying quasilocal algebra and $\omega = \omega \circ \alpha_t$. Then we can define an automorphism $\bar{\alpha}_t$ on the Weyl algebra $\mathcal{W}_\omega$ by

$$\bar{\alpha}_t W_\omega(C) = W_\omega(\alpha_t C). \tag{22}$$

If the state is sufficiently clustering and the automorphism sufficiently local then it has been shown [8], [12] that

$$\lim e^{iH_N t} e^{i \frac{\sum (C^k - \omega(C^k))}{\sqrt{N}}} e^{-iH_N t} = \bar{\alpha}_t W_\omega(C).$$

This means that for an invariant state $\bar{\alpha}_t \circ \hat{\alpha}_t = id$. But already in [9] it was shown that for mean field theories this does not hold in general. This is even of more interest because it has consequences in experiments. In [10] on the basis of this evolution mesoscopic entanglement was produced. In [11] this experiment was analyzed in the framework of the fluctuation algebra.

Since the state ω can change on the quasilocal level it is clear that we have to look for a generalization of (22), i.e. the evolution of the fluctuations. We expect that we can formulate the evolution as an isomorphism between the fluctuation algebras corresponding to different times. Therefore we introduce the following map:

Definition 3.

$$\tilde{\alpha}_t W_{\omega_t}(C) = w\text{-}\lim e^{iH_N t} e^{i \sum_{j=1}^N \frac{C_j - \omega_t(C_j)}{\sqrt{N}}} e^{-iH_N t}. \tag{23}$$

Now the "weak" limit (where the limit is taken in the sense of (17)) has to be taken in ω and it has to be shown that this limit exists. Since the correction term $\omega_t(C_j)$ is taken in ω_t we start with an expression that is well defined in $\mathcal{W}_{\omega_t}$ and is mapped by $\tilde{\alpha}_t$ into an operator in $\mathcal{W}_\omega$. Again under the assumptions of [8] and [12] this map reduces to $\tilde{\alpha}_t W_{\omega_t}(C) = W_\omega(\bar{\alpha}_t C)$. But in [9] we had examples where the state is invariant on the quasilocal level but the state on the fluctuation algebra changes in time. Nevertheless the result in [8], [12] encourages us to define

Definition 4.

$$\tilde{\alpha}_t W_{\omega_t}(A) = W_\omega(\alpha_t^* A). \tag{24}$$

Here we only assume that α_t^* exists as a linear map on an appropriate subset of the quasilocal algebra that in the special examples has to be specified. Obviously α_t^* must satisfy $\omega \circ \alpha_t = \omega \circ \alpha_t^*$. Finally we can consider

Definition 5.

$$\alpha_t^0 = \tilde{\alpha}_t \circ \hat{\alpha}_t. \tag{25}$$

This is a map from $\mathcal{W}_\omega$ into $\mathcal{W}_\omega$. Under the assumption that α_t^* exists we have $\alpha_t^0 W_\omega(C) = W_\omega(\alpha_t^* \alpha_{-t} C)$. In our example α_t^0 will turn out to be actually an automorphism but not the identity as in [8], [12].

To show that the limit (23) and consequently also (25) exists in our models (1) and (2) we examine first formally the corresponding evolution equations on the fluctuation algebra, e.g. the derivative of (23) $\tilde{\alpha}_t(S_k)$,

$$\begin{aligned} \frac{d}{dt} S_k(t) &= \frac{d}{dt} \lim \sum_{j=1}^{N} e^{iH_N^2 t} \frac{\sigma_k^j - \omega(\sigma_k^j)}{\sqrt{N}} e^{-iH_N^2 t} \\ &= \lim i \left[H_N, \sum_{j=1}^{k} \frac{\sigma_k^j}{\sqrt{N}} \right] - \frac{d}{dt} \frac{\omega(\sigma_k^j)}{\sqrt{N}}. \end{aligned} \tag{26}$$

As a first control we realize that the differential equation for the commutation relations demands

$$i \left[H_N, \sum_{j=1}^{k} \frac{\sigma_k^j}{\sqrt{N}} \right] \frac{\sum \sigma_l^j}{\sqrt{N}} + i \left[H_N, \sum_{j=1}^{k} \frac{\sigma_l^j}{\sqrt{N}} \right] \frac{\sum \sigma_k^j}{\sqrt{N}} = \frac{d}{dt} \omega([\sigma_k^j, \sigma_l^j]), \tag{27}$$

which in fact is satisfied if we use the evolution equation of the quasilocal state. The similar condition has to hold for τ. Finally we argue that the differential equation can be integrated to an automorphism group (25).

More explicitly we calculate for the time evolution determined by (2)

$$\frac{d}{dt}S_z = \lim \frac{d}{dt}\frac{\sum \sigma_z^j - Ns_z}{\sqrt{N}}$$
$$= \lim \left(-\frac{\sum \sigma_y^j \sum \tau_x^k}{N\sqrt{N}} + \frac{\sum \sigma_x^j \sum \tau_x^k}{N\sqrt{N}} + \sqrt{N}s_y a_x - \sqrt{N}s_x t_y \right)$$
$$= a_y S_x - a_x S_y - s_y T_x + s_x T_y. \tag{28}$$

Here the clustering property of the quasilocal state is essential and guarantees that the term proportional to $\sqrt{N}$ cancels. Similarly

$$\frac{d}{dt}S_x = \epsilon S_y - a_y S_z - s_z T_y,$$
$$\frac{d}{dt}S_y = -\epsilon S_x + a_x S_z + s_z T_x,$$
$$\frac{d}{dt}T_z = a_x S_y - a_y S_x + s_y T_x - s_x T_y,$$
$$\frac{d}{dt}T_x = -a_z S_y - s_y T_z,$$
$$\frac{d}{dt}T_y = a_z S_x + s_x T_z, \tag{29}$$

or for the laser system

$$\frac{d}{dt}S_z = a_y S_x - a_x S_y - s_y A_x + s_x A_y,$$
$$\frac{d}{dt}S_x = \epsilon S_y - a_y S_z - s_z A_y,$$
$$\frac{d}{dt}S_y = -\epsilon S_x + a_x S_z + s_z A_x,$$
$$\frac{d}{dt}A_x = -S_y - kA_y,$$
$$\frac{d}{dt}A_y = S_x + kA_x. \tag{30}$$

Note that the quadratic evolution equation of the quasilocal state turns into a linear evolution equation on the fluctuation algebra. It is a map between the Weyl algebras because the commutation relations on the derivation level can be controlled via

$$\frac{d}{dt}[S_x, S_y] = \frac{d}{dt}s_z = \left[\frac{d}{dt}S_x, S_y\right] + \left[S_x, \frac{d}{dt}S_y\right] = s_x a_y - s_y a_x \tag{31}$$

and similarly for the other commutation relations, e.g.

$$\begin{aligned} \frac{d}{dt}[S_x, S_z] &= -\frac{d}{dt}s_y = -\epsilon s_x + s_z a_x, \\ \frac{d}{dt}[A_x, A_y] &= [S_y, A_y] + [S_x, A_y] = 0. \end{aligned} \tag{32}$$

It remains to argue that the differential equation defines automorphisms (25). First we take α_t^* in (24) to be the same linear map on the σ, τ, A_x, A_y as the one corresponding to the differential equations (28), (29) resp. (30). We observe

Lemma.

$$\omega(\alpha_t \sigma_k) = \omega(\alpha_t^* \sigma_k) \tag{33}$$

and similarly for the other expressions.

Proof. (28), (29), (30) are homogeneous differential equations for operators with expectation value 0. This implies that they satisfy e.g.

$$\frac{d}{dt}\omega(\tilde{\alpha}_t S_k) = 0 = \frac{d}{dt}\omega(\alpha_t^* \sigma_k - \alpha_t \sigma_k),$$

where we take α_t^* to be determined by (28), (29), (30).

To demonstrate the existence of the limit in (25) we calculate

$$\begin{aligned} &\lim_{N\to\infty} \omega\left(e^{i\vec{b}\frac{\sum(\vec{\sigma}-\omega_t(\vec{\sigma}))}{\sqrt{N}}} e^{iH_N t} e^{i\vec{a}\frac{\sum(\alpha_t^*\vec{\sigma}-\omega_t(\alpha_t^*\vec{\sigma}))}{\sqrt{N}}} e^{-iH_N t}\right) - \omega(e^{i\vec{b}\vec{S}} e^{i\vec{a}\vec{S}}) \\ &= \lim \int_0^t dt' \omega\left(e^{i\vec{b}\frac{\sum(\vec{\sigma}-\omega_t(\vec{\sigma}))}{\sqrt{N}}} e^{iH_N t'} e^{i\alpha\vec{a}\frac{\sum(\alpha_{t'}^*\vec{\sigma}-\omega_{-t'}(\vec{\sigma}))}{\sqrt{N}}} \right. \\ &\quad \circ \left\{ i\left[H_N, \vec{a}\frac{\sum(\alpha_{t'}^*\vec{\sigma} - \omega_{-t'}(\vec{\sigma}))}{\sqrt{N}}\right] - \sqrt{N}\frac{d}{dt'}\vec{a}(\alpha_{t'}^*\vec{\sigma} - \omega_{-t'}(\vec{\sigma}))\right\} \\ &\quad \left. \circ\, e^{i(1-\alpha)\vec{a}\frac{\sum(\vec{\sigma}-\omega_{t'}(\vec{\sigma}))}{\sqrt{N}}} e^{-iH_N t'}\right). \end{aligned} \tag{34}$$

We can control how to commute

$$e^{i(1-\alpha)\vec{a}\frac{\sum(\vec{\sigma}-\omega_{t'}(\vec{\sigma}))}{\sqrt{N}}}$$

through, since it acts linearly. Therefore we only have to estimate the commutator [,] with its correction term. We can apply our result on the quasilocal level where of course we have to take into account that (16) converges sufficiently fast because now we have an additional scaling like $\sqrt{N}$. All remaining operators are unitaries and do not effect the strong convergence to 0. In the same way we can replace σ by τ or

$$\vec{a}\frac{\sum(\vec{\sigma}-\omega(\vec{\sigma}))}{\sqrt{N}}$$

by $a_x A_x + a_y A_y$. We collect the result in

Theorem 3. *Under the assumption on the state* (3), (4) *with the Hamiltonian* (2) *resp.* (3), (6), (7) *and the Hamiltonian* (1) *the time evolution of the fluctuations determined by* (23) *satisfies the differential equations* (28), (29) *resp.* (30). *Combined with the natural map between the fluctuation algebras induced by the evolution of the quasilocal state* (25) *the time evolution corresponds to an automorphism on the fluctuation algebra*, *that is not trivial.*

In this way we obtained a time evolution that in fact is an automorphism on the fluctuation algebras of the σ- and the τ-field respectively of the laser field together with the fluctuation algebra of the σ-field. By taking the corresponding conditional expectation values we can reduce the time automorphism on the tensor product of the Weyl algebras to a completely positive map on the individual factors. This seems to be natural especially for the laser algebra, if we take the observation of the fluctuation algebra of the σ field as being outside of our experimental facilities. This positive map will in general not be an automorphism so that the entropy of the laser field need not be constant in time. Whether it increases or decreases or fluctuates depends on the details of the underlying quasilocal state.

4. Stationary and Periodic States

The time evolution of the quasilocal states admits invariant states as well as periodic states. We cannot assume that an arbitrary state converges to a limit state. This may be forbidden by the constants of the motion. But for given constants of the motions we can at least find periodic states. These preferred states can be stable or unstable under small perturbations. We will study how the time evolution of the fluctuation algebra looks like in these special states and whether we can find some relations with the stability of the quasilocal states.

We are essentially interested in the long time behavior of the quasilocal state and how the fluctuations are effected. According to the estimates in [4] the limit of the reliability of the time evolution of the quasilocal state is determined by the size of the fluctuations. Since the evolution equation on the fluctuation algebra is linear the fluctuations either are periodic or they depend linearly or exponentially on time. Therefore the time scale on which the evolution of the quasilocal state is reliable based on the estimates in [4] is in general of the order $\ln N$ but is longer for periodic fluctuations. Therefore we expect – or hope – that the fluctuations of the stable quasilocal states behave periodically.

In fact we will show that the stability behavior on the quasilocal level and on the level of the fluctuation algebra is the same, at least in the simplest examples.

Example 1. We will assume in the following that ϵ and k in the Hamiltonian in $H_N^{1,2}$ vanishes, which corresponds to the fact that the interaction between the fields is dominating. Then only those states can be invariant in time for which $s_x = s_y = 0$, $a_x = a_y = 0$. s_z respectively s_z, a_z can be arbitrary. If however $s_y = a_y = 0$ whereas $s_x, a_x \neq 0$ but small compared to s_z, a_z, then s_y, a_y will evolve in the same direction or in different directions, depending on the relative directions of s_z, a_z and s_x, a_x so that their contribution to $(d/dt)s_z \sim -s_z a_x^2 + s_x^2 a_z$ annihilates or adds up depending whether $s_z a_z > 0$ or < 0. Though in general this annihilation will not be complete we can consider states with the same sign of a_z, s_z as more stable than the others. (If we consider H_N^1 this corresponds to take $a_z = 1$.)

If we turn now to the fluctuation algebra in this invariant state then the equations reduce to

$$\frac{d^2}{dt^2} S_x = -s_z \frac{d}{dt} T_y = -s_z a_z S_x. \tag{35}$$

As we expected the fluctuations in fact rotate if we are in a stable state where $a_z s_z > 0$ but increase exponentially if we are in an unstable state where $a_z s_z < 0$.

Example 2. As another typical solution we consider the quasilocal state periodic in time for which

$$\frac{d}{dt} s_z = \frac{d}{dt} a_z = 0,$$

$$s_x = s \cos \nu t, \quad s_y = s \sin \nu t, \quad a_x = a \cos \nu t, \quad a_y = a \sin \nu t,$$

which is a solution of the equation of motion if

$$\nu = \frac{as_z - \epsilon s}{s} = \frac{a_z s}{a}. \tag{36}$$

(For H_N^1 we take $a_z = 1$. Now also in this model the photon number is constant.) Again to simplify our calculations we only consider $\epsilon = 0$. Then we observe that necessarily $s_z a_z > 0$ or for the laser model $1 > s^2 a^{-2} = s_z > 0$. The evolution equation for the fluctuation algebra becomes for the evolution corresponding to H_N^1

$$\frac{d}{dt} S_z = sA_y \cos \nu t - sA_x \sin \nu t - aS_y \cos \nu t + aS_x \sin \nu t. \tag{37}$$

We define

$$\begin{aligned} S_1 &= S_x \cos \nu t + S_y \sin \nu t, \qquad & S_2 &= -S_x \sin \nu t + S_y \cos \nu t, \\ A_1 &= A_x \cos \nu t + A_y \sin \nu t, \qquad & A_2 &= -A_x \sin \nu t + A_y \cos \nu t. \end{aligned} \tag{38}$$

This is in fact the desired isomorphism $\hat{\alpha}_t$ between the fluctuation algebras. We can verify

$$\begin{aligned} [A_1, A_2] &= i, \\ [S_z, S_1] &= i \sin \nu t \cos \nu t - i \sin \nu t \cos \nu t = 0, \\ [S_z, S_2] &= is \sin^2 \nu t + is \cos^2 \nu t = is, \\ [S_1, S_2] &= i(\cos^2 \nu t + \sin^2 \nu t) s_z = is_z, \end{aligned}$$

so that all c numbers that appear in the commutation relations are invariant in time and S_1, S_2, A_1, A_2 evolve in the same Weyl algebra. With these new variables the differential equation reduces to

$$\frac{d}{dt} S_z = sA_2 - aS_2, \quad \frac{d}{dt} S_1 = -s_z A_2 + \nu S_2, \quad \frac{d}{dt} S_2 = aS_z + s_z A_1 - \nu S_1,$$

$$\frac{d}{dt} A_1 = -S_2 + \nu A_2, \qquad \frac{d}{dt} A_2 = S_1 - \nu A_1, \tag{39}$$

which combines to

$$\begin{aligned} \frac{d^2}{dt^2} S_1 &= (-s_z - \nu^2) S_1 + 2 s_z \nu A_1 + a \nu S_z, \\ \frac{d^2}{dt^2} A_1 &= -(\nu^2 + s_z) A_1 + 2 \nu S_1 - a S_z, \end{aligned}$$

$$\frac{d^2}{dt^2}S_2 = (-a^2 - s_z - \nu^2)S_2 + (2s_z\nu + as)A_2,$$
$$\frac{d^2}{dt^2}A_2 = -(\nu^2 + s_z)A_2 + 2\nu S_2. \tag{40}$$

Thus we have reduced the differential equation to one with constant parameters that correspond to rotation frequencies $\bar{\nu}$ of the fluctuation algebra given by $\bar{\nu}^4 - \bar{\nu}^2(a^2 + 4s^2/a^2) = 0$ where we have used (36) together with $a_z = 1$. Therefore one frequency $\bar{\nu}_1^2 = a^2 + 4s^2/a^2$ gives a rotation whereas the other a linear increase of the fluctuations. If the differential equation is based on H_N^2 then again $S_z + T_z$ is constant in time and though the time derivative of T_2 contains T_z finally again only S_2 and T_2 are coupled in the equation that determines the rotation frequency. This shows that both Hamiltonians lead to a qualitatively similar time evolution both on the level of the quasilocal algebra as for the fluctuation algebra.

We can assign to the time evolution of the fluctuation algebra combined with the laser algebra an effective Hamiltonian

$$H = -2as(A_2S_2 + A_1S_1) + a^2(S_1^2 + S_2^2 + S_z^2) + s^2(A_1^2 + A_2^2)$$
$$= (aS_1 - sA_1)^2 + (aS_2 - sA_2)^2 + a^2S_z^2, \tag{41}$$

which demonstrates once more that we succeeded to reduce the problem to automorphisms, but due to our special choice of $\hat{\alpha}_t$ these automorphisms satisfy even the group property.

Analyzing the spectrum of the Hamiltonian we observe that it consists of a harmonic oscillator plus an operator with absolutely continuous positive spectrum so that the total spectrum is positive and absolutely continuous. This corresponds on the basis of the evolution equation that the frequency $\bar{\nu}_2 = 0$ cannot be suppressed by initial conditions and on the basis of the fluctuation algebra that the state changes in time.

Again we wonder whether the corresponding time evolution of the quasilocal state is stable under perturbations. We write $s_z = s\cos\delta$, $s_x = s\sin\delta\sin\phi$, $s_y = s\sin\delta\cos\phi$, $a_x = a\sin\xi$, $a_y = a\cos\xi$ where s is constant in time whereas $a = a(\delta)$. The evolution equations (9) in these variables read

$$\sin\delta\frac{d}{dt}\delta = -s\sin\delta\sin\phi\cos\xi + s\sin\delta\cos\phi\sin\xi = s\sin\delta\sin(\phi - \xi),$$
$$s\cos\delta\sin\phi\frac{d}{dt}\delta + s\sin\delta\cos\phi\frac{d}{dt}\phi = -as\cos\delta\cos\xi,$$
$$s\cos\delta\cos\phi\frac{d}{dt}\delta - s\sin\delta\sin\phi\frac{d}{dt}\phi = as\cos\delta\sin\xi,$$

which combines to

$$\sin\delta \frac{d}{dt}\phi = -a\cos\delta\cos(\phi-\xi),$$

$$\frac{da}{dt}\sin\xi + a\cos\xi\frac{d}{dt}\xi = -s\sin\delta\cos\xi,$$
$$\frac{da}{dt}\cos\xi - a\sin\xi\frac{d}{dt}\xi = s\sin\delta\sin\xi.$$

In addition we have

$$a\frac{d}{dt}\xi = -s\sin\delta\cos(\phi-\xi)$$

and therefore

$$a\sin\delta\frac{d}{dt}(\phi-\xi) = (-a^2\cos\delta + s\sin^2\delta)\cos(\phi-\xi).$$

Here $(d/dt)(\phi-\xi) = 0$ if $s\sin^2\delta_0 - a^2\cos\delta_0 = 0$ which corresponds to the periodic solution, where $(d/dt)\delta = 0$. If we perturb δ slightly then

$$\begin{aligned}\frac{d^2}{dt^2}\delta &= -s\cos^2(\phi-\xi)(-a^2\cos\delta + s\sin^2\delta)\\ &= -s\cos^2(\phi-\xi)\sin\delta_0(a^2+2s\cos\delta_0)(\delta-\delta_0)\end{aligned}$$

so that the solution is in fact stable under perturbations of the possible δ_0 that corresponds to the permitted value $s_z > 0$.

5. The Entropy Balance

Since it was necessary to introduce the fluctuation algebra of the atoms in order to describe the evolution of the laser as an automorphism, the evolution reduced to the laser alone is only a completely positive map that may increase or decrease the entropy. In the special situation when $s_x = s_y = a_x = a_y = 0$ when the quasilocal state is invariant this does not yet imply that the state over the laser is invariant. We start with a product state over two Weyl algebras. If we are in the stable situation then the two Weyl algebras rotate. For a special state over both Weyl algebras the rotation is not felt by the state and it is invariant in time. But in general the state will be periodic. If however the quasilocal state is unstable the fluctuations increase exponentially and the state changes.

In the case when s_z is constant and the quasilocal state rotates around the z-axes then we have seen that the solution of the fluctuation algebra contains the frequency 0 which means that we have a linear increase of the evolution $A_1(t) \sim A_1 + tS_2$. If we therefore evaluate, ignoring the periodic part

$$\operatorname{tr} \rho_A \otimes \rho_S \circ \alpha_t e^{i\gamma A_1} \sim \operatorname{tr} \rho_A e^{i\bar{\gamma} A_1 - \bar{\gamma}^2 ct^2},$$

which shows that in the course of time $\omega(A^*A)$ increases whereas $\omega(A) = 0$. Therefore the entropy of the state over the laser algebra increases, which corresponds to the fact that information moves into the correlations between the laser and the fluctuations.

6. Conclusion

We have studied the time evolution of two level atoms interacting with a laser in an appropriate thermodynamic limit. Based on the results in [4] we extended the time evolution of the quasilocal state to the fluctuation algebra as defined in [8]. This extension is necessary if one wants to keep the total entropy fixed. We observed that even when the state on the quasilocal level does not change in time for the local algebra on the level of the fluctuation algebra it changes producing increasing correlations between the fluctuations. In the controlled examples the amount how the fluctuations change was related whether the underlying quasilocal state is stable or unstable under small perturbations, in the stable situation the fluctuations change at most linearly in time, otherwise exponentially.

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