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Ring-Like Structures Corresponding to MV-Algebras via Symmetric Difference*

By

Ivan Chajda and **Helmut Länger**

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Abstract

The well-known natural bijective correspondence between Boolean algebras and Boolean rings is generalized from Boolean algebras to MV-algebras. The ring-like structures arising this way correspond in a natural bijective manner to so-called strong De Morgan algebras.

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1. Introduction

The fundamental operations in logic are disjunction, conjunction and negation. Describing them algebraically leads to lattice-like structures with some sort of complementation. In the classical case one obtains Boolean algebras which correspond to Boolean rings in a natural bijective way. Ring addition and ring multiplication can be logically interpreted as exclusive disjunction and conjunction, respectively.

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The natural bijective correspondence between Boolean algebras and Boolean rings was generalized from Boolean algebras to orthomodular lattices in [9] and [17], to ortholattices in [2], to bounded lattices with an involutory antiautomorphism in [10], to pseudocomplemented semilattices in [4], to generalized orthomodular lattices in [6] and to generalized ortholattices in [5]. In [3] the interplay between the structures introduced in [2] and [10] is investigated. The structures introduced in [10] were studied in more detail in [11]–[15].

The interplay between lattice distributivity on the one side and associativity of ring addition and ring distributivity on the other side is of importance when applying the considered structures in the foundations of axiomatic quantum mechanics. E.g., the fact that in the associativity law only four operation symbols are involved whereas the number of operation symbols occurring in the distributivity law is five has some physical meaning concerning the complexity of the corresponding physical experiments.

2. MV-Algebras

First we introduce the notion of an MV-algebra. These algebras provide an adequate semantics for the infinite-valued Łukasiewicz logic (cf. [1]).

Definition 2.1. An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms:

(MV1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $\neg\neg x = x$,
(MV5) $x \oplus 1 = 1$ and
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

where 1 denotes the element $\neg 0$.

Example 2.1. $(B, \vee, ', 0)$, where $(B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, is an MV-algebra.

Example 2.2. $([0, 1], \oplus, \neg, 0)$, where $x \oplus y := \min(x + y, 1)$ and $\neg x := 1 - x$ for all $x, y \in [0, 1]$, is an MV-algebra.

Lemma 2.1. *In an MV-algebra* $(A, \oplus, \neg, 0)$ *it holds* $x \oplus \neg x = 1$ *for all* $x \in A$.

Proof. For all $x \in A$ we have

$$x \oplus \neg x = \neg x \oplus x = \neg(\neg 1 \oplus x) \oplus x = \neg(\neg x \oplus 1) \oplus 1 = 1. \qquad \square$$

In every MV-algebra $(A, \oplus, \neg, 0)$ a binary operation $\odot$ is defined by

$$x \odot y := \neg(\neg x \oplus \neg y)$$

for all $x, y \in A$. In $(B, \vee, {}', 0)$, where $(B, \vee, \wedge, {}', 0, 1)$ is a Boolean algebra, $\odot = \wedge$ and in $([0,1], \oplus, \neg, 0)$ defined in Example 2.2, $x \odot y = \max(x + y - 1, 0)$ for all $x, y \in [0,1]$.

The following result is well-known (cf. [1]).

Proposition 2.1. If $(A, \oplus, \neg, 0)$ is an MV-algebra and one defines

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$

and

$$x \wedge y := \neg(\neg(x \oplus \neg y) \oplus \neg y)$$

for all $x, y \in A$ then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice the corresponding partial order of which is given by $x \leq y$ if and only if $\neg x \oplus y = 1$ $(x, y \in A)$.

For more results on MV-algebras we refer to the monographs [7] and [16].

3. Pseudorings

We now want to associate suitable ring-like structures to MV-algebras. In this context it should be mentioned that connections between MV-algebras and semirings were investigated in [8].

Definition 3.1. A *pseudoring* is an algebra $(R, +, \cdot, 1)$ of type $(2, 2, 0)$ satisfying the following axioms:

(P1) $(xy)z = x(yz)$,
(P2) $xy = yx$,
(P3) $x1 = x$,
(P4) $1 + (1 + x) = x$,
(P5) $x0 = 0$,
(P6) $(1 + x(1 + y))(1 + y) = (1 + y(1 + x))(1 + x)$ and
(P7) $1 + (1 + x(1 + y))(1 + y(1 + x)) = x + y$

where 0 denotes the element $1 + 1$.

Remark 3.1. Axioms (P2) and (P7) imply the commutativity of $+$.

The following theorem establishes a natural bijective correspondence between MV-algebras and pseudorings:

Theorem 3.1. *If A is a set then the formulas*

$$x + y := \neg(x \oplus \neg y) \oplus \neg(\neg x \oplus y),$$
$$xy := \neg(\neg x \oplus \neg y)$$

and

$$x \oplus y := 1 + (1 + x)(1 + y),$$
$$\neg x := 1 + x$$

for all $x, y \in A$ induce mutually inverse bijections between the set of all MV-algebras on A and the set of all pseudorings on A.

Remark 3.2. It holds $x + y = (x \odot \neg y) \oplus (\neg x \odot y)$ for all $x, y \in A$. Hence in $(B, \vee, {}', 0)$, where $(B, \vee, \wedge, {}', 0, 1)$ is a Boolean algebra, $+$ coincides with the symmetric difference and in $([0, 1], \oplus, \neg, 0)$ defined in Example 2.2, $+$ coincides with the distance of real numbers.

Proof of Theorem 3.1. If $(A, \oplus, \neg, 0)$ is an MV-algebra and $x + y := \neg(x \oplus \neg y) \oplus \neg(\neg x \oplus y)$ and $xy := \neg(\neg x \oplus \neg y)$ for all $x, y \in A$ then

$$
\begin{aligned}
&1 + x = \neg(1 \oplus \neg x) \oplus \neg(\neg 1 \oplus x) = \neg x,\\
&1 + 1 = \neg 1 = 0,\\
&(xy)z = \neg(\neg\neg(\neg x \oplus \neg y) \oplus \neg z) = \neg((\neg x \oplus \neg y) \oplus \neg z)\\
&\qquad = \neg(\neg x \oplus (\neg y \oplus \neg z)) = \neg(\neg x \oplus \neg\neg(\neg y \oplus \neg z)) = x(yz),\\
&xy = \neg(\neg x \oplus \neg y) = \neg(\neg y \oplus \neg x) = yx,\\
&x1 = \neg(\neg x \oplus \neg 1) = x,\\
&1 + (1 + x) = \neg\neg x = x,\\
&x0 = \neg(\neg x \oplus \neg 0) = 0,\\
&(1 + x(1 + y))(1 + y) = (\neg(x(\neg y)))(\neg y)\\
&\quad = \neg(\neg\neg\neg(\neg x \oplus \neg\neg y) \oplus \neg\neg y)\\
&\quad = \neg(\neg(\neg x \oplus y) \oplus y) = \neg(\neg(\neg y \oplus x) \oplus x)\\
&\quad = \neg(\neg\neg\neg(\neg y \oplus \neg\neg x) \oplus \neg\neg x)\\
&\quad = (\neg(y(\neg x)))(\neg x) = (1 + y(1 + x))(1 + x),\\
&1 + (1 + x(1 + y))(1 + y(1 + x)) = \neg((\neg(x(\neg y)))(\neg(y(\neg x))))\\
&\quad = \neg\neg(\neg\neg\neg(\neg x \oplus \neg\neg y) \oplus \neg\neg\neg(\neg y \oplus \neg\neg x))\\
&\quad = \neg(\neg x \oplus y) \oplus \neg(\neg y \oplus x) = x + y
\end{aligned}
$$

and

$$1+(1+x)(1+y) = \neg((\neg x)(\neg y)) = \neg\neg(\neg\neg x \oplus \neg\neg y) = x \oplus y$$

for all $x, y, z \in A$.

If, conversely, $(A, +, \cdot, 1)$ is a pseudoring and $x \oplus y := 1+(1+x)(1+y)$ and $\neg x := 1+x$ for all $x, y \in A$ then

$$\begin{aligned}
&\neg 0 = 1+0 = 1+(1+1) = 1,\\
&(x \oplus y) \oplus z = 1+(1+(1+(1+x)(1+y)))(1+z)\\
&\quad = 1+((1+x)(1+y))(1+z) = 1+(1+x)((1+y)(1+z))\\
&\quad = 1+(1+x)(1+(1+(1+y)(1+z))) = x \oplus (y \oplus z),\\
&x \oplus y = 1+(1+x)(1+y) = 1+(1+y)(1+x) = y \oplus x,\\
&x \oplus 0 = 1+(1+x)(1+0) = 1+(1+x)1 = 1+(1+x) = x,\\
&\neg\neg x = 1+(1+x) = x,\\
&x \oplus 1 = 1+(1+x)(1+(1+0)) = 1+(1+x)0 = 1+0 = 1,\\
&\neg(\neg x \oplus y) \oplus y = 1+(1+(1+(1+(1+(1+x))(1+y))))(1+y)\\
&\quad = 1+(1+x(1+y))(1+y) = 1+(1+y(1+x))(1+x)\\
&\quad = 1+(1+(1+(1+(1+(1+y))(1+x))))(1+x)\\
&\quad = \neg(\neg y \oplus x) \oplus x,\\
&\neg(x \oplus \neg y) \oplus \neg(\neg x \oplus y)\\
&\quad = 1+(1+(1+(1+(1+x)(1+(1+y)))))\\
&\qquad \times (1+(1+(1+(1+(1+x))(1+y))))\\
&\quad = 1+(1+(1+x)y)(1+x(1+y)) = x+y
\end{aligned}$$

and

$$\neg(\neg x \oplus \neg y) = 1+(1+(1+(1+x))(1+(1+y))) = xy$$

for all $x, y, z \in A$. □

4. Strong De Morgan Algebras

In this section we want to show that MV-algebras correspond in a natural bijective way to certain algebras that are similar to so-called De Morgan algebras. This yields also a natural bijective correspondence between these so-called strong De Morgan algebras and pseudorings.

Definition 4.1. A *De Morgan algebra* is an algebra $(A, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $'$ is an antiendomorphism of $(A, \vee, \wedge)$. A *strong De Morgan* *algebra* is an algebra $(A, \vee, \wedge, ({}^a; a \in A), 0, 1)$ where $(A, \vee, \wedge, 0, 1)$ is a bounded lattice, for each $a \in A$, $([a, 1], \vee, \wedge, {}^a, a, 1)$ is a De Morgan algebra, $(x^y)^y = x$ for all $x, y \in A$ with $x \geq y$ and

$$(x \vee (y \vee z)^z)^{(y \vee z)^z} = (y \vee (x \vee z)^z)^{(x \vee z)^z}$$

for all $x, y, z \in A$.

Theorem 4.1. *If A is a set then the formulas*

$$\begin{aligned} x \vee y &:= \neg(\neg x \oplus y) \oplus y, \\ x \wedge y &:= \neg(\neg(x \oplus \neg y) \oplus \neg y), \\ x^a &:= \neg x \oplus a \end{aligned}$$

and

$$\begin{aligned} x \oplus y &:= (x^0 \vee y)^y, \\ \neg x &:= x^0 \end{aligned}$$

for all $x, y, a \in A$ induce mutually inverse bijections between the set of all MV-algebras on A and the set of all strong De Morgan algebras on A.

Proof. First let $(A, \oplus, \neg, 0)$ be an MV-algebra and put $x \vee y := \neg(\neg x \oplus y) \oplus y$, $x \wedge y := \neg(\neg(x \oplus \neg y) \oplus \neg y)$ and $x^a := \neg x \oplus a$ for all $x, y, a \in A$. It is well-known that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice the corresponding partial order relation of which is given by $x \leq y$ if and only if $\neg x \oplus y = 1$ $(x, y \in A)$. Moreover, for x, y, $a \in A$

$$\begin{aligned} &a \leq x^a = \neg x \oplus a \quad \text{since} \quad \neg a \oplus \neg x \oplus a = 1, \\ &a \leq x \leq y \quad \text{implies} \quad \neg(\neg y \oplus a) \oplus (\neg x \oplus a) = \neg(\neg y \oplus a) \oplus a \oplus \neg x \\ &\qquad = \neg(\neg a \oplus y) \oplus y \oplus \neg x = 1, \quad \text{i.e.} \quad y^a \leq x^a, \end{aligned}$$

and

$$a \leq x \quad \text{implies} \quad \neg(\neg x \oplus a) \oplus a = \neg(\neg a \oplus x) \oplus x = x, \quad \text{i.e. } (x^a)^a = x.$$

Put $x \circ y := (x \vee y)^y$ for all $x, y \in A$. Then

$$x \circ y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = \neg(\neg y \oplus \neg x \oplus y) \oplus \neg x \oplus y = \neg x \oplus y$$

for all $x, y \in A$ and hence

$$\begin{aligned} &(x \vee (y \vee z)^z)^{(y \vee z)^z} = x \circ (y \circ z) = \neg x \oplus \neg y \oplus z = \neg y \oplus \neg x \oplus z \\ &\qquad = y \circ (x \circ z) = (y \vee (x \vee z)^z)^{(x \vee z)^z}, \\ &(x^0 \vee y)^y = \neg(\neg(\neg\neg x \oplus y) \oplus y) \oplus y = \neg(\neg y \oplus x \oplus y) \oplus x \oplus y = x \oplus y \end{aligned}$$

and

$$x^0 = \neg x \oplus 0 = \neg x$$

for all $x, y, z \in A$.

Conversely, if $(A, \vee, \wedge, ({}^a; a \in A), 0, 1)$ is a strong De Morgan algebra and $x \oplus y := (x^0 \vee y)^y$, $\neg x := x^0$ and $x \circ y := (x \vee y)^y$ for all $x, y \in A$ then

$$\begin{aligned}
&x \circ (y \circ z) = y \circ (x \circ z),\\
&(x \circ 0) \circ 0 = x^{00} = x,\\
&x \oplus y = (x \circ 0) \circ y,\\
&x \oplus y = (x \circ 0) \circ y = (x \circ 0) \circ ((y \circ 0) \circ 0) = (y \circ 0) \circ ((x \circ 0) \circ 0)\\
&\qquad = (y \circ 0) \circ x = y \oplus x,\\
&(x \oplus y) \oplus z = z \oplus (x \oplus y) = (z \circ 0) \circ ((x \circ 0) \circ y)\\
&\qquad\qquad = (x \circ 0) \circ ((z \circ 0) \circ y) = x \oplus (z \oplus y) = x \oplus (y \oplus z),\\
&x \oplus 0 = (x \circ 0) \circ 0 = x,\\
&\neg\neg x = x^{00} = x,\\
&x \oplus 1 = (x^0 \vee 1)^1 = 1,\\
&\neg x \oplus y = ((x^0)^0 \vee y)^y = (x \vee y)^y,\\
&\neg(\neg x \oplus y) \oplus y = ((x \vee y)^y \vee y)^y = ((x \vee y)^y)^y = x \vee y
\end{aligned}$$

and hence

$$\begin{aligned}
&\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,\\
&\neg(\neg(x \oplus \neg y) \oplus \neg y) = \neg(\neg(\neg\neg x \oplus \neg y) \oplus \neg y) = (x^0 \vee y^0)^0 = x \wedge y
\end{aligned}$$

and

$$\neg x \oplus a = (x \vee a)^a = x^a \quad \text{if} \quad x \geq a$$

for all $x, y, z, a \in A$. □

Combining Theorems 3.1 and 4.1 yields

Theorem 4.2. *If A is a set then the formulas*

$$\begin{aligned}
x \vee y &:= 1 + (1 + x(1 + y))(1 + y),\\
x \wedge y &:= (1 + (1 + x)y)y,\\
x^a &:= 1 + x(1 + a)
\end{aligned}$$

and

$$x + y := ((x^0 \vee y^0)^{y^0} \vee ((x \vee y)^y)^0)^{((x\vee y)^y)^0},$$
$$xy := ((x \vee y^0)^{y^0})^0$$

for all x, y, $a \in A$ *induce mutually inverse bijections between the set of all pseudorings on A and the set of all strong De Morgan algebras on A.*

As an immediate consequence of Theorems 3.1, 4.1 and 4.2 and their proofs we obtain

Corollary 4.1. *If* $(R, +, \cdot, 1)$ *is a pseudoring and*

$$(R, \vee, \wedge, (^a; a \in R), 0, 1)$$

denotes the corresponding strong De Morgan algebra then (i) *and* (ii) *hold*:

(i) $a \leq b$ *if and only if* $a(1 + b) = 0$

(ii) $(R, \vee, \wedge, {}^0, 0, 1)$ *is a Boolean algebra if and only if* $(1 + xx)x = 0$ *for every* $x \in R$.

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Authors' addresses: Ivan Chajda, Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 77900 Olomouc, Czech Republic, E-Mail: chajda@inf.upol.cz; Helmut Länger, Institute of Discrete Mathematics and Geometry, Research Unit Algebra, Vienna University of Technology, Wiedner Hauptstraße 8–10, 1040 Vienna, Austria, E-Mail: h.laenger@tuwien.ac.at