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Sequences of Iterates of Random-Valued Vector Functions and Solutions of Related Equations

By

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Abstract

Given a probability space $(\Omega, \mathcal{A}, P)$ and a separable metric space X we obtain some theorems on the existence of solutions $\varphi: X \to \mathbb{R}$ of equations of the form $\varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega)$. They are given via probability distribution of the limit of the sequence of iterates of the given function $f: X \times \Omega \to X$.

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1. Introduction

Equations of the form

$$\varphi(x) = \int_\Omega \varphi(L(\omega)x + M(\omega))P(d\omega) \tag{1}$$

and their generalizations, e.g.,

$$\varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega), \tag{2}$$

appear in many branches of mathematics and solutions φ of them are extensively studied (see [2; Part 4] and [1; Part 3]). Very important results on the existence and uniqueness of their solutions have been proved by G. Derfel in [6], by J. Morawiec in [12], [13] and by K. Baron and W. Jarczyk in [3]. In [3] and [8] the iterates of random-valued functions were used (for the first time in [3]) to solve equations of form (2) and (1) on the unit interval and in Banach spaces, respectively. The aim of the present paper is to obtain suitable results on metric spaces for Eq. (2).

2. Random-Valued Functions and Their Iterates

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a separable metric space X. Let $\mathcal{B}(X)$ denote the σ-algebra of all Borel subsets of X. We say that $f\colon X\times\Omega \to X$ is a random-valued function if it is measurable with respect to the product σ-algebra $\mathcal{B}(X)\otimes\mathcal{A}$. The iterates of such a function f are defined by

$$f^1(x,\omega_1,\omega_2,\ldots) = f(x,\omega_1),$$
$$f^{n+1}(x,\omega_1,\omega_2,\ldots) = f(f^n(x,\omega_1,\omega_2,\ldots),\omega_{n+1})$$

for x from X and $(\omega_1,\omega_2,\ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n\colon X\times\Omega^\infty \to X$ is a random-valued function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the n-th iterate f^n is $\mathcal{B}(X)\otimes\mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all the sets of the form

$$\{(\omega_1,\omega_2,\ldots)\in\Omega^\infty\colon (\omega_1,\omega_2,\ldots,\omega_n)\in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [4], [9]; also [10; Sec. 1.4].) These iterates were defined independently in [4] and [7]. In [4] and [9] some conditions are established which guarantee the convergence (a.s. and in L^1) of $(f^n(x,\cdot))$.

Fix a random-valued function $f\colon X\times\Omega \to X$.

According to [8; Theorem 5.1], if the sequence of iterates $(f^n(x,\cdot))$ of f converges in measure and the limit does not depend on x, then every continuous and bounded solution $\varphi\colon X \to \mathbb{R}$ of (2) is constant. We will prove that the probability distribution of the limit of $(f^n(x,\cdot))$ always produces a bounded solution of (2) which is in addition nonconstant provided the limit really depends on x; cf. also [3; Propositions 2.1 and 2.2]. By this we can get a nonconstant solution of (2) also in the case where it has no nonconstant continuous solutions.

3. The Case of the Almost Sure Convergence

We start with an observation that if the metric space X is complete, then the set

$$\{(x,\omega)\in X\times\Omega^\infty\text{: the sequence } (f^n(x,\omega)) \text{ converges in } X\}$$

belongs to the σ-algebra $\mathcal{B}(X)\otimes\mathcal{A}^\infty$. Hence for every Borel set $B\subset X$ the set

$$\{(x,\omega)\in X\times\Omega^\infty\text{: the sequence } (f^n(x,\omega)) \text{ converges in } X \text{ and its limit belongs to } B\})$$

belongs to this σ-algebra too. Therefore we can define the function $\pi\colon X\times\mathcal{B}(X)\to[0,1]$ putting

$$\pi(x,B) = P^\infty(\{\omega\in\Omega^\infty\text{: the sequence } (f^n(x,\omega)) \text{ converges in } X \text{ and its limit belongs to } B\}). \tag{3}$$

According to the Fubini theorem for every $B\in\mathcal{B}(X)$ the function $\pi(\cdot,B)$ is Borel. It is easily seen that for every $x\in X$ the function $\pi(x,\cdot)$ is a measure.

The following theorem is an extension of [3; Proposition 2.1].

Theorem 1. *If X is a separable and complete metric space, then for every Borel and bounded function $u\colon X\to\mathbb{R}$ the function $\varphi\colon X\to\mathbb{R}$ defined by*

$$\varphi(x) = \int_X u(y)\pi(x,dy)$$

is a Borel solution of (2).

Proof. The fact that φ is Borel follows from [11; Sec. 8, Theorem 13 on p. 141]. Now we are going to prove that it is a solution of (2). To get this let us note first that for all $x\in X$ and $B\in\mathcal{B}(X)$ we have

$$\begin{aligned}\pi(x,B) &= P^\infty(\{\omega\in\Omega^\infty\text{: the sequence } (f^{n+1}(x,\omega)) \text{ converges in } X \\ &\qquad \text{and its limit belongs to } B\}) \\ &= P^\infty(\{(\omega_1,\omega_2,\ldots)\in\Omega^\infty\text{: the sequence } (f^n(f(x,\omega_1),\omega_2,\ldots)) \\ &\qquad \text{converges in } X \text{ and its limit belongs to } B\}) \\ &= \int_\Omega P^\infty(\{(\omega_2,\omega_3,\ldots)\in\Omega^\infty\text{: the sequence } (f^n(f(x,\omega_1),\omega_2,\ldots)) \\ &\qquad \text{converges in } X \text{ and its limit belongs to } B\})P(d\omega_1) \\ &= \int_\Omega \pi(f(x,\omega_1),B)P(d\omega_1).\end{aligned}$$

From this and from [11; Sec. 8, Theorem 13 on p. 141] it follows that

$$\int_\Omega \varphi(f(x,\omega))P(d\omega) = \int_\Omega \left(\int_X u(y)\pi(f(x,\omega),dy)\right)P(d\omega)$$
$$= \int_X u(y)\pi(x,dy) = \varphi(x). \qquad \square$$

Since

$$\int_X \mathbf{1}_B(y)\pi(x,dy) = \pi(x,B)$$

for all $x\in X$ and $B\in\mathcal{B}(X)$, the above theorem ensures the existence of nonconstant Borel solutions of (2) in the case where the limit of $(f^n(x,\cdot))$ is nonconstant (with respect to x); more precisely: if $x,y\in X$ and $\pi(x,\cdot)\neq\pi(y,\cdot)$, then for some $B\in\mathcal{B}(X)$ the function $\pi(\cdot,B)$ is a nonconstant Borel solution of (2).

4. The Case of the Convergence in Measure

If the sequence $(f^n(x,\cdot))$ converges a.s. to a measurable function $\xi(x,\cdot)\colon\Omega^\infty\to X$, then the function $\pi(x,\cdot)$ given by (3) is a probability measure and

$$\pi(x,B) = P^\infty(\xi(x,\cdot)\in B). \tag{4}$$

In the case of the convergence in measure we have the following theorem.

Theorem 2. *Suppose that X is a separable metric space and for every $x\in X$ the sequence $(f^n(x,\cdot))$ converges in measure to a measurable function $\xi(x,\cdot)$ and define $\pi(x,\cdot)\colon\mathcal{B}(X)\to[0,1]$ by* (4). *Then*:

(i) *for every continuous and bounded function $u\colon X\to\mathbb{R}$ the function $\varphi\colon X\to\mathbb{R}$ defined by*

$$\varphi(x) = \int_X u(y)\pi(x,dy) = \int_{\Omega^\infty} u(\xi(x,\omega))P^\infty(d\omega)$$

is a bounded solution of (2);

(ii) *if $x,y\in X$ and $\pi(x,\cdot)\neq\pi(y,\cdot)$, then Eq.* (2) *has a bounded solution $\varphi\colon X\to\mathbb{R}$ such that $\varphi(x)\neq\varphi(y)$.*

Proof. Fix $x,y\in X$. Since for every $\omega_1\in\Omega$ the sequence $(u(f^n(f(x,\omega_1),\cdot)))$ converges in measure to the function

$u(\xi(f(x,\omega_1),\cdot))$, by the Lebesgue-Vitali dominated Theorem we obtain

$$\lim_{n\to\infty}\int_{\Omega^\infty} u(f^{n+1}(x,\omega_1,\omega_2,\ldots))P^\infty(d(\omega_2,\omega_3,\ldots))$$
$$= \int_{\Omega^\infty} u(\xi(f(x,\omega_1),\omega_2,\omega_3,\ldots))P^\infty(d(\omega_2,\omega_3,\ldots)) = \varphi(f(x,\omega_1)).$$

Moreover, on account of the Fubini theorem, for every $n \in \mathbb{N}$ the function (of the variable $\omega_1 \in \Omega$)

$$\int_{\Omega^\infty} u(f^{n+1}(x,\omega_1,\omega_2,\omega_3,\ldots))P^\infty(d(\omega_2,\omega_3,\ldots))$$

is measurable. From this it follows that the function $\varphi \circ f(x,\cdot)$: $\Omega \to \mathbb{R}$ is measurable and

$$\int_\Omega \varphi(f(x,\omega))P(d\omega)$$
$$= \lim_{n\to\infty}\int_\Omega \left(\int_{\Omega^\infty} u(f^{n+1}(x,\omega_1,\omega_2,\ldots))P^\infty(d(\omega_2,\omega_3,\ldots))\right)P(d\omega_1)$$
$$= \lim_{n\to\infty}\int_{\Omega^\infty} u(f^n(x,\omega_1,\omega_2,\ldots))P^\infty(d(\omega_1,\omega_2,\ldots))$$
$$= \int_{\Omega^\infty} u(\xi(x,\omega))P^\infty(d\omega) = \varphi(x).$$

To get the second part of our theorem it is enough to observe that if for every continuous function u: $X \to \mathbb{R}$ we have

$$\int_X u(z)\pi(x,dz) = \int_X u(z)\pi(y,dz),$$

then (see, e.g., [5; Theorem 1.1.3]) $\pi(x,\cdot) = \pi(y,\cdot)$. □

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