

## More on Rootless Matrices

By

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YOOD in [5] considered the problem of finding “rootless” matrices, i.e., complex  $n \times n$ -matrices  $A$  such that there is no matrix  $S$  and no positive integer  $r \geq 2$  such that  $S^r = A$ .

Among others it was shown there that any nilpotent  $n \times n$ -matrix  $A$  ( $A^t = 0$  for some  $t \geq 1$ ) with maximal rank ( $n - 1$ ) is rootless. In connection with this result the question was raised, whether there are nilpotent matrices of rank less than  $n - 1$  which still are rootless.

Using results from [1, Kapitel 8.6, 8.7] (English version [2]) we will show that there are nilpotent rootless matrices of all (reasonable) ranks.

The main tool we will use is the concept of JORDAN normal forms [1, Kapitel 7], [2, chapter 7]. Compare also [3], where even the infinite-dimensional case is covered.

Let  $\lambda$  be a complex number and  $n$  a positive integer. Then the  $n \times n$ -matrix

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

is called a *Jordan block* of length  $n$  associated with the eigenvalue  $\lambda$ . Note that  $J_n(\lambda)$  is regular, if  $\lambda \neq 0$ , and nilpotent, if  $\lambda = 0$ . For short we write  $J_n$  for  $J_n(0)$ :  $J_n := J_n(0)$ .

We say that two  $n \times n$ -matrices  $A$  and  $B$  are *similar* ( $A \sim B$ ) if there is some regular matrix  $T$  such that  $T^{-1}AT = B$ . Obviously similarity of matrices is an equivalence relation. The equivalence classes are called *similarity classes*.

If  $A_1, A_2, \dots, A_k$  are square matrices, the matrix

$$A := \text{diag}(A_1, A_2, \dots, A_k) := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

is called a *block diagonal matrix*.

**Remark 1.** For any  $n \times n$ -matrix  $A$  there are complex numbers  $\lambda_1, \lambda_2, \dots, \lambda_k$  and positive integers  $n_1, n_2, \dots, n_k$  such that  $A$  is similar to the block diagonal matrix (the *Jordan normal form* of  $A$ )

$$J := \text{diag}(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \dots, J_{n_k}(\lambda_k)).$$

Moreover, the normal forms  $J$  and

$$J' := \text{diag}(J_{m_1}(\mu_1), J_{m_2}(\mu_2), \dots, J_{m_l}(\mu_l))$$

are similar if, and only if,  $k = l$  and if there is some permutation  $\pi$  such that  $\mu_i = \lambda_{\pi(i)}$  and  $m_i = n_{\pi(i)}$  for all  $i = 1, 2, \dots, k$ .

The problem of finding  $m$ -th roots  $S$  of  $A$ , i.e., solutions of  $S^m = A$ , is invariant under similarity: If  $B = T^{-1}AT$ , then  $S^m = A$  if, and only if,  $(T^{-1}ST)^m = B$ .

Thus for finding (all) rootless matrices it is enough to investigate Jordan normal forms. Moreover, by the following lemma, the consideration of nilpotent matrices is all we need.

**Lemma 1.** *Let  $A = \text{diag}(R, N)$ , where  $R$  is invertible and  $N$  nilpotent. Then  $A$  is rootless if, and only if,  $N$  is rootless.*

*Proof.* It is well known ([1, Kapitel 8.6], [4, pp. 96–97]), that any regular matrix has roots of any order. If, for example,  $R = J_n(\lambda)$  and  $\lambda \neq 0$ , then  $J_n(\lambda) = \lambda(E_n + \lambda^{-1}J_n)$ . For any  $\mu$  with  $\mu^m = \lambda$  the matrix

$$S := \mu \sum_{j=0}^{n-1} \binom{1/m}{j} \lambda^{-j} J_n^j$$

satisfies  $S^m = R$ . (Note that  $J_n^n = 0$  and that  $(1+x)^{1/m} = \sum_{j=0}^{\infty} \binom{1/m}{j} x^j$  for  $|x| < 1$ .) If the normal form of  $R$  contains more

than one block, by the above procedure one may construct  $m$ -th roots for all of them (all blocks have to be regular). The block diagonal matrix built from those roots is an  $m$ -th root of the normal form similar to  $R$ .

Now suppose that  $W$  is an  $m$ -th root of  $A$ . Let us write  $W$  in the form of a block matrix

$$W = \begin{bmatrix} U_1 & V_1 \\ V_2 & U_2 \end{bmatrix}$$

with square matrices  $U_1, U_2$  of dimensions equal to the dimension of  $R$  and  $N$ , respectively. Since  $W^m = A$  the matrix  $W$  commutes with  $A$ :  $AW = WA$ . This implies in particular  $V_1N = RV_1$ . We want to show that  $V_1 = 0$ . (Similar arguments then also show that  $V_2 = 0$ .) Let  $N^k = 0$ . Multiplying the equation above by  $N^{k-1}$  from the right gives  $0 = V_1N^k = RV_1N^{k-1}$  which, by the regularity of  $R$  implies  $V_1N^{k-1} = 0$ . Repeated use of the equation above then leads to  $V_1N = 0$  and finally to  $V_1 = 0$ .

Thus  $W = \text{diag}(U_1, U_2)$ . This implies that  $W^m = \text{diag}(U_1^m, U_2^m) = A$  if, and only if,  $U_1^m = R$  and  $U_2^m = N$ . This gives the desired result, since  $U_1^m = R$  always has solutions.  $\square$

In [5] it was shown that the matrix

$$J_2 = J_2(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is rootless. It was also mentioned there that the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is (non-nilpotent) rootless. The above lemma explains this quite easily since this matrix equals  $\text{diag}(J_1(1), J_2(0))$ .

Next we investigate the normal form of  $J_n^m$ , where

$$J_n := J_n(0) := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Let  $e_1, e_2, \dots, e_n$  be the column vectors of the canonical base in  $\mathbb{C}^n$ . Defining  $e_i := 0$  if  $i \leq 0$  then implies  $J_n e_i = e_{i-1}$  and (by induction)

$J_n^m e_i = e_{i-m}$  for all  $i$  and  $m$ . This can be used to find the normal form of  $J_n^m$  [1, Kapitel 8.7]. Write  $n = km + r$  with nonnegative  $k$  and  $0 \leq r < m$ . Then each  $t$ ,  $1 \leq t \leq n$ , has a unique representation in the form  $t = lm + j$ , now with  $1 \leq j \leq m$ . In the case  $1 \leq j \leq r$  the quotient  $l$  lies in the range  $0, 1, \dots, k$ . If  $r + 1 \leq j \leq m$  the range of  $l$  is  $0, 1, \dots, k - 1$ .

Since  $J_n^m e_{lm+j} = e_{(l-1)m+j}$  this means that the Jordan normal form of  $J_n^m$  contains  $r$  blocks  $J_{k+1}$  and  $m - r$  blocks  $J_k$  (and no others):

$$J_n^m \sim \text{diag}(\underbrace{J_{k+1}, \dots, J_{k+1}}_{r\text{-times}}, \underbrace{J_k, \dots, J_k}_{(m-r)\text{-times}}).$$

More explicitly we may state that

$$P_{n,m}^{-1} J_n^m P_{n,m} = \text{diag}(\underbrace{J_{k+1}, \dots, J_{k+1}}_{r\text{-times}}, \underbrace{J_k, \dots, J_k}_{(m-r)\text{-times}}),$$

where the regular  $n \times n$ -matrix  $P_{n,m}$  has the column vectors

$$\begin{array}{cccccc} e_1 & e_{m+1} & e_{2m+1} & \cdots & e_{(k-1)m+1} & e_{km+1} \\ e_2 & e_{m+2} & e_{2m+2} & \cdots & e_{(k-1)m+2} & e_{km+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e_r & e_{m+r} & e_{2m+r} & \cdots & e_{(k-1)m+r} & e_{km+r} \\ e_{r+1} & e_{m+r+1} & e_{2m+r+1} & \cdots & e_{(k-1)m+r+1} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ e_m & e_{m+m} & e_{2m+m} & \cdots & e_{(k-1)m+m} & \end{array}$$

Using this we also are able to determine the Jordan normal form of  $J^m$  for arbitrary  $J := \text{diag}(J_{v_1}, J_{v_2}, \dots, J_{v_s})$ . Since  $J^m = \text{diag}(J_{v_1}^m, J_{v_2}^m, \dots, J_{v_s}^m)$  we see that

$$J^m \sim \text{diag}(\underbrace{J_{k_1+1}, \dots, J_{k_1+1}}_{r_1\text{-times}}, \underbrace{J_{k_1}, \dots, J_{k_1}}_{(m-r_1)\text{-times}}, \dots, \underbrace{J_{k_s+1}, \dots, J_{k_s+1}}_{r_s\text{-times}}, \underbrace{J_{k_s}, \dots, J_{k_s}}_{(m-r_s)\text{-times}}),$$

where  $v_i = k_i m + r_i$  with  $0 \leq r_i < m$  for all  $i = 1, 2, \dots, s$ .

Summarizing we have

**Theorem 1.** *Let  $N$  be a nilpotent  $n \times n$ -matrix such that*

$$N \sim \text{diag}(\underbrace{J_1, \dots, J_1}_{\mu_1\text{-times}}, \underbrace{J_2, \dots, J_2}_{\mu_2\text{-times}}, \dots, \underbrace{J_n, \dots, J_n}_{\mu_n\text{-times}}),$$

where  $\mu_1, \mu_2, \dots, \mu_n \geq 0$  and  $\sum_{i=1}^n i \mu_i = n$ .

1. Then  $N$  has an  $m$ -th root, if, and only if, there is some  $s \geq 1$  and there are some  $v_1 = k_1m + r_1, v_2 = k_2m + r_2, \dots, v_s = k_sm + v_s$  ( $k_l \geq 0, 0 \leq r_l < m$ ) such that

$$\mu_i = \sum_{l \in L_i} r_l + \sum_{l \in L'_i} (m - r_l), \quad i = 1, 2, \dots, n,$$

where  $L_i := \{l \mid k_l + 1 = i\}$  and  $L'_i := \{l \mid k_l = i\}$ .

2. If  $N$  has an  $m$ -th root then the number  $\sum \mu_i$  of blocks in its normal form has to be a multiple of  $m$ .

(Compare [1, 3].)

This theorem, in principle, is all one needs to determine all nilpotent rootless matrices of a given order  $n$ . But because of its general and combinatorial nature which makes heavy use of solutions of systems of diophantine equations it seems to be no adequate tool for determining all possible ranks of rootless nilpotent  $n \times n$ -matrices.

**Theorem 2.** Let  $n \geq 2$ . Then there is a rootless nilpotent  $n \times n$ -matrix of rank  $\rho$  if, and only if,  $2 \leq \rho \leq n - 1$ . For any  $2 \leq \rho \leq n - 1$  the matrix

$$N_\rho := \text{diag}(J_{\rho+1}, \underbrace{J_1, \dots, J_1}_{(n-\rho-1)\text{-times}})$$

is nilpotent, has rank  $\rho$ , and is rootless.

Moreover, for all  $1 \leq \rho \leq n - 2$  there are nilpotent  $n \times n$ -matrices of rank  $\rho$  which are not rootless.

*Proof.* For all  $2 \leq m \leq n - 1$  the rank of  $J_n^m$  is  $n - m$ . This proves the second part of the theorem.

Any nilpotent matrix of rank 1 is similar to

$$N_2 := \text{diag}(J_2, \underbrace{J_1, \dots, J_1}_{(n-2)\text{-times}}),$$

which is similar to  $J_n^{n-1}$ . Thus there is no rootless nilpotent matrix of rank 1.

Let  $2 \leq \rho \leq n - 1$ . Obviously the rank of  $N_\rho$  is  $\rho$ . Thus we (only) must show that  $N_\rho$  is rootless. Suppose that  $B^m = N_\rho$  for some  $B$ . Then  $B$  is similar to  $J := \text{diag}(J_{v_1}, J_{v_2}, \dots, J_{v_s})$  for some positive  $s$  and some positive  $v_1, v_2, \dots, v_s$  with  $v_1 + v_2 + \dots + v_s = n$ . Writing  $v_l = k_lm + r_l$  ( $0 \leq r_l < m$ ) we get by the preceding theorem that  $J^m$  is similar to a Jordan normal form which, for each  $l$ , contains  $r_l$  blocks  $J_{k_l+1}$  and  $m - r_l$  blocks  $J_{k_l}$ . But  $J^m$  is similar to  $B^m = N_\rho$ . Thus  $N_\rho$  also contains those blocks.

Note that  $N_\rho$  contains exactly one block  $J_{\rho+1}$  and exactly  $n - \rho - 1$  blocks  $J_1$  (and no others). Since  $N_\rho$  contains one block  $J_{\rho+1}$ , there must be some  $l$  such that  $\rho + 1 = k_l$  or  $\rho + 1 = k_l + 1$ . The latter possibility cannot appear. In this case  $N_\rho$  would also contain at least  $m - r_l$  blocks  $J_{k_l} = J_\rho$ . Since  $m - r_l > 0$ , there would be at least one block of that kind. But  $1 < \rho < \rho + 1$ , so this is impossible. Accordingly  $\rho + 1 = k_l$ . Moreover  $N_\rho$  contains no block  $J_{\rho+2}$ . Thus  $r_l = 0$ . Since  $J_{\rho+1}$  appears only once as a block in  $N_\rho$  we conclude that  $m = m - r_l \leq 1$ . On the other hand  $m - r_l > 0$ . Thus  $m = m - r_l = 1$ . This means that  $N_\rho$  has no nontrivial root.  $\square$

### References

- [1] GANTMACHER, F. R. (1986) *Matrizentheorie*. Springer, Berlin-Heidelberg-New York
- [2] GANTMACHER, F. R. (1998) *The Theory of Matrices*, Vol. 1 (Transl. from the Russian by HIRSCH, K. A.) Reprint of the 1959 Translation. AMS Chelsea, Providence, RI
- [3] NG, C. T., ZHANG, W.-N. (to appear) *Cyclic Decomposition and Solutions of an Iterative Functional Equation*
- [4] WEDDERBURN, J. H. M. (1934) *Lectures on Matrices*. Amer. Math. Soc. Coll. Publ., Vol. 17. Providence, RI
- [5] YOOD, B. (2002) Rootless matrices. *Math. Mag.* **75**: 219–223

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