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Inner Symmetries for Moebius Maps

By

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Abstract

Differentiable isomorphisms of Moebius systems are considered (SCHWEIGER [2]). A map is called an inner symmetry if it commutes with the map T of the Moebius system and permutes the cells of the time-1-partition. This notion is discussed for Moebius systems with two and three branches. An extension to 2-dimensional cases is outlined.

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0. Introduction

Definition. Let B be an interval and T: $B \to B$ be a map. We assume that there is a countable collection of intervals $(J_k), k \in I, \#I \geq 2$ and an associated sequence of matrices

$$\alpha(k) = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix},$$

$a_k d_k - b_k c_k \neq 0$, with the properties:

- $\bigcup_{k \in I} \overline{J_k} = \overline{B}$, $J_m \cap J_n = \emptyset$ if $n \neq m$.
- $Tx = \dfrac{c_k + d_k x}{a_k + b_k x}, \qquad x \in J_k.$
- $T|_{J_k}$ is a bijective map from J_k onto B.

Then we call (B, T) a Moebius system.

This is a special case of a fibred system (SCHWEIGER [1]). Since $T|_{J_k}$ is bijective the inverse map $V_k: B \to J_k$ exists. The corresponding matrix will be denoted by $\beta(k)$. We denote furthermore

$$\omega(k;x) := |V_k'(x)| = \frac{|a_k d_k - b_k c_k|}{(d_k - b_k x)^2}.$$

Then a nonnegative measurable function h is the density of an invariant measure iff $h(x) = \sum_{k \in I} h(V_k x)\omega(k;x)$.

Remark. It is easy to see that we can assume $B = [a,b]$, $B = [a,\infty[$ or $B =]-\infty, b]$ but $B = \mathbb{R}$ is excluded (since $\#I \geq 2$).

Definition. The Moebius system (B^*, T^*) is called a *natural dual* of (B,T) if there is a partition $\{I_k^*\}$ such that

$$T^* y = \frac{b_k + d_k y}{a_k + c_k y},$$

i.e., the matrix $\alpha^*(k)$ is the transposed matrix of $\alpha(k)$.

In the paper SCHWEIGER [2] the following definition was given.

Definition. The Moebius system (B^*, T^*) is *differentiably isomorphic* to (B,T) if there is a map $\psi: B \to B^*$ such that ψ' exists almost everywhere and the commutativity condition $\psi \circ T = T^* \circ \psi$ holds.

What I had in mind was a more precise definition, namely for all $k \in I$ the commutativity condition

$$\psi \circ \alpha(k) = \alpha(k)^* \circ \psi \tag{1}$$

should hold and this property was used in all what followed in the paper. However, one might ask if it is possible that

$$\psi \circ \alpha(k) = \alpha(\pi k)^* \circ \psi \tag{2}$$

holds for a permutation $\pi: I \to I$ of the index set.

1. The Case of Two Branches

It is well known that in the case of two branches, namely $\#I = 2$ the system (B^*, T^*) is always differentiably isomorphic to (B,T) in the sense of Eq. (1). We take $B = [0,1]$ with $c = \frac{1}{2}$ as the midpoint of the partition. We put $I = \{\lambda, \mu\}$ and there are four subtypes $(\varepsilon_1, \varepsilon_2)$ where $\varepsilon_j = 1$ stands for an increasing map and $\varepsilon_j = -1$ for a decreasing map. Now we ask for the possibility of satisfying (2) with exchanging λ and μ

$$\begin{aligned}\varphi \circ \alpha(\lambda) &= \alpha(\mu)^* \circ \varphi,\\ \varphi \circ \alpha(\mu) &= \alpha(\lambda)^* \circ \varphi.\end{aligned} \tag{3}$$

Since $\psi \circ \alpha(k) = \alpha(k)^* \circ \psi,\ k \in \{\lambda, \mu\}$, is always satisfied, we obtain

$$\psi^{-1} \circ \varphi \circ \alpha(\lambda) = \psi^{-1} \circ \alpha(\mu)^* \circ \varphi = \alpha(\mu) \circ \psi^{-1} \circ \varphi.$$

Therefore the system (B, T) allows an *inner symmetry* $\chi = \psi^{-1} \circ \varphi$ such that

$$\chi \circ V(\lambda) = V(\mu) \circ \chi. \tag{4}$$

Conversely, if (B, T) allows such an inner symmetry, we can find a differentiable map φ which satisfies (3), namely $\varphi = \psi \circ \chi$. Therefore it is enough to solve Eq. (4).

Theorem 1. (*a*) *For types* (1, 1) *and* (−1, −1) *there are infinitely many solutions of Eq.* (4).

(*b*) For types (1, −1) *and* (−1, 1) *no solution exists.*

Proof. The only Moebius transformation which permutates the points $\{0, \frac{1}{2}, 1\}$ in an appropriate way is $\chi(t) = 1 - t$.

It is easy to see that the matrices $\alpha(\lambda)$ and $\alpha(\mu)$ depend on one free parameter. The conditions on λ and μ are necessary to avoid attractive fixed points or poles in the domain of definition.

(a) Type $(1,1),\ 0 < \lambda \leq 1,\ \mu \leq 0$:

$$\alpha(\lambda) = \begin{pmatrix} \lambda & 1-2\lambda \\ 0 & 1 \end{pmatrix}, \quad \alpha(\mu) = \begin{pmatrix} \mu & -1-\mu \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \lambda & 1-2\lambda \\ 0 & 1 \end{pmatrix}$$
$$= \rho \begin{pmatrix} \mu & -1-\mu \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

From this we see

$$\lambda = \frac{1}{1-\mu} = -\rho.$$

Note that $0 < \lambda \leq 1$ is consistent with $\mu \leq 0$.

Type $(-1,-1)$, $\lambda<2$, $\mu<2$:

$$\alpha(\lambda)=\begin{pmatrix}1 & -\lambda\\ 1 & -2\end{pmatrix},\qquad \alpha(\mu)=\begin{pmatrix}\mu & 2-2\mu\\ 2 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -\lambda\\ 1 & -2\end{pmatrix}$$
$$=\rho\begin{pmatrix}\mu & 2-2\mu\\ 2 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}.$$

This gives the conditions

$$\rho=\frac{1}{2-\mu},$$
$$\lambda=\frac{2-2\mu}{2-\mu}.$$

(b) For type $(1,-1)$ the branch $\alpha(\lambda)$ has the fixed point $x=0$ but $x=1$ is not a fixed point for $\alpha(\mu)$. A similar reasoning excludes the type $(-1,1)$.

2. The Case of Three Branches

As in SCHWEIGER [2] we consider the partition $0<\frac{1}{2}<\frac{2}{3}<1$ and three maps depending on parameters λ,μ,ν, say. By continuity reasons the only case of satisfying (2) with a proper permutation is the exchange of λ and ν which leaves μ fixed,

$$\psi\circ\alpha(\lambda)=\alpha(\nu)^{\#}\circ\psi,$$
$$\psi\circ\alpha(\mu)=\alpha(\mu)^{\#}\circ\psi. \tag{5}$$

In a similar way we can also define an inner symmetry χ as a Moebius map such that

$$\chi\circ\alpha(\lambda)=\alpha(\nu)\circ\chi,$$
$$\chi\circ\alpha(\mu)=\alpha(\mu)\circ\chi. \tag{6}$$

We will consider inner symmetries first. We will see that for three branches an inner symmetry in almost all cases leads to a solution of (5).

The only Moebius transformation which exchanges $\{0,\frac{1}{2},\frac{2}{3},1\}$ is

$$\chi(t)=\frac{2-2t}{2-t}.$$

Looking at the cases in which either $x=0$ or $x=1$ is a fixed point we can exclude the types $(1,1,-1)$, $(1,-1,-1)$, $(-1,1,1)$, and $(-1,-1,1)$. For the remaining types we find the following result.

Theorem 2. *All types* $(1,1,1)$, $(1,-1,1)$, $(-1,1,-1)$, $(-1,-1,-1)$ *allow an inner symmetry for an infinite set of parameters* λ *and* ν.

Proof. We first list the six matrices $\alpha(k)$, $k = \lambda, \mu, \nu$, which will be used in our calculations.

$$\varepsilon_1 = 1, \begin{pmatrix} \lambda & 1-2\lambda \\ 0 & 1 \end{pmatrix},$$
$$\varepsilon_1 = -1, \begin{pmatrix} -1 & -\lambda+2 \\ -1 & 2 \end{pmatrix},$$
$$\varepsilon_2 = 1, \begin{pmatrix} 2\mu-1 & 2-3\mu \\ -1 & 2 \end{pmatrix},$$
$$\varepsilon_2 = -1, \begin{pmatrix} \mu-2 & 3-2\mu \\ -2 & 3 \end{pmatrix},$$
$$\varepsilon_3 = 1, \begin{pmatrix} \nu-2 & -\nu+3 \\ -2 & 3 \end{pmatrix},$$
$$\varepsilon_3 = -1, \begin{pmatrix} 2\nu-1 & 1-3\nu \\ -1 & 1 \end{pmatrix}.$$

(a) Type $(1,1,1)$, $0<\lambda \le 1$, $0<\mu$, $1 \le \nu$. We consider first the equation $\chi \circ \alpha(\mu) = \alpha(\mu) \circ \chi$.

$$\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2\mu-1 & 2-3\mu \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\mu-1 & 2-3\mu \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}.$$

This gives $\mu = \frac{1}{2}$ as the unique solution for the middle branch. The further equations

$$\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} \lambda & 1-2\lambda \\ 0 & 1 \end{pmatrix} = \rho\begin{pmatrix} \nu-2 & -\nu+3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}$$

give the relation $\lambda\nu = 1$ which is compatible with the results in SCHWEIGER [2].

(b) Type $(-1,1,-1)$, $0<\lambda$, $0<\mu$, $0<\nu$. In a similar way we find from

$$\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} -1 & -\lambda+2 \\ -1 & 2 \end{pmatrix} = \rho\begin{pmatrix} 2\nu-1 & 1-3\nu \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}$$

the relation $4\lambda\nu = 1$.

(c) Type $(1,-1,1)$, $0<\lambda\leq 1$, $0<\mu$, $1\leq\nu$. We need only consider the equation $\chi\circ\alpha(\mu)=\alpha(\mu)\circ\chi$,

$$\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} \mu-2 & 3-2\mu \\ -2 & 3 \end{pmatrix}=\begin{pmatrix} \mu-2 & 3-2\mu \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}.$$

This gives the unique value $\mu=1$. As before the other equations show $\lambda\nu=1$.

(d) Type $(-1,-1,-1), 0<\lambda, 0<\mu, 0<\nu$. This gives the conditions $\mu=1$ and $4\lambda\nu=1$.

Remark. The invariant density $h=h(x)$ satisfies the equation $h(x)=h(\chi(x))|\chi'(x)|$. If ξ_λ, ξ_μ, ξ_ν are the fixed points of T then clearly we find $\chi(\xi_\lambda)=\xi_\nu$, $\chi(\xi_\mu)=\xi_\mu$, and $\chi(\xi_\nu)=\xi_\lambda$.

Theorem 3. (a) *For type* $(1,1,1)$ *there is no solution of Eq.* (2).
(b) *For types* $(1,-1,1),(-1,1,-1),(-1,-1,-1)$ *there are infinitely many solutions.*

Proof. Type $(1,1,1)$: For the existence of an inner symmetry we found the relation $\lambda\nu=1$ and $\mu=\frac{1}{2}$. A differentiably isomorphic dual (Eq. (1)) exists if the condition

$$2\lambda\mu+2\nu=\lambda\nu+\lambda$$

is satisfied. This leads to $2\nu=1$ which is not allowed.

Type $(1,-1,1)$: Here we found $\lambda\nu=1$ and $\mu=1$. A differentially isomorphic dual exists if the condition

$$\lambda\nu=\mu$$

holds. Therefore any inner symmetry leads to a solution of (5) but not vice versa.

Type $(-1,1,-1)$: Here we had $\mu=\frac{1}{2}$ and $4\lambda\nu=1$. The condition for an isomorphic dual is given as

$$2\lambda\mu+\mu=2\lambda\nu+\lambda$$

which is satisfied.

Type $(-1,-1,-1)$: In this case $\mu=1$ and $4\lambda\nu=1$. The condition

$$4\lambda\nu+\lambda+\nu=\mu\nu+\lambda\mu+\mu$$

is also satisfied.

3. Outlook: 2-Dimensional Generalizations

These ideas can be extended to higher dimensions. We will consider three 2-dimensional algorithms. The kernel

$$K(x_1, x_2, y_1, y_2) = \frac{1}{(1 + x_1 y_1 + x_2 y_2)^3}$$

leads to a dual system with transposed matrices.

Brun Algorithm. We consider the map on $B = \{(x_1, x_2)\colon 0 \le x_2 \le x_1 \le 1\}$

$$T(x_1, x_2) = \begin{cases} \left(\dfrac{x_1}{1 - x_1}, \dfrac{x_2}{1 - x_1}\right), & x_1 \le \frac{1}{2}, \\ \left(\dfrac{1 - x_1}{x_1}, \dfrac{x_2}{x_1}\right), & x_1 + x_2 \le 1, \\ \left(\dfrac{x_2}{x_1}, \dfrac{1 - x_1}{x_1}\right), & 1 \le x_1 + x_2. \end{cases}$$

Then the dual algorithm T^* exists on the set $B^* = \{(y_1, y_2)\colon 0 \le y_1, 0 \le y_2 \le 1\}$. The map

$$\psi(s, t) = \left(\frac{1 - s}{s}, \frac{t}{s}\right)$$

gives a differentiable isomorphism. Since the point $x = (0, 0)$ is the only fixed point on the boundary it is seen quite easily that no inner symmetry can be found.

Selmer Algorithm. Here we consider the set $D = \{(x_1, x_2)\colon 1 \le x_1 + x_2, 0 \le x_2 \le x_1 \le 1\}$ and the map

$$T(x_1, x_2) = \begin{cases} \left(\dfrac{1 - x_2}{x_1}, \dfrac{x_2}{x_1}\right), & x_2 \le \frac{1}{2}, \\ \left(\dfrac{x_2}{x_1}, \dfrac{1 - x_2}{x_1}\right), & \frac{1}{2} \le x_2. \end{cases}$$

The dual exists on the set $D^* = \{(y_1, y_2)\colon 0 \le y_1, 0 \le y_2\}$ and the map

$$\psi(s, t) = \left(\frac{-1 + s + t}{1 - s}, \frac{s - t}{1 - s}\right)$$

is a differentiable isomorphism. Again, since $x = (1, 0)$ is the only fixed point on the boundary an inner symmetry is not allowed.

Parry-Daniels Map. We use the projected version on the triangle $\Delta = \{(x_0, x_1)\colon\ 0 \le x_0,\ 0 \le x_1,\ x_0 + x_1 \le 1\}$. The partition is labelled by permutations on the indices $0, 1, 2$ and the map is given piecewise as follows,

$$\begin{aligned}
T_\varepsilon(x_0, x_1) &= \left(\frac{x_0}{1 - x_0 - x_1}, \frac{-x_0 + x_1}{1 - x_0 - x_1}\right),\\
T_{(01)}(x_0, x_1) &= \left(\frac{x_1}{1 - x_0 - x_1}, \frac{x_0 - x_1}{1 - x_0 - x_1}\right),\\
T_{(12)}(x_0, x_1) &= \left(\frac{x_0}{x_1}, \frac{1 - 2x_0 - x_1}{x_1}\right),\\
T_{(021)}(x_0, x_1) &= \left(\frac{1 - x_0 - x_1}{x_1}, \frac{-1 + 2x_0 + x_1}{x_1}\right),\\
T_{(02)}(x_0, x_1) &= \left(\frac{1 - x_0 - x_1}{x_0}, \frac{-1 + 2x_0 + x_1}{x_0}\right),\\
T_{(012)}(x_0, x_1) &= \left(\frac{x_1}{x_0}, \frac{1 - x_0 - 2x_1}{x_0}\right).
\end{aligned}$$

There also exists a natural dual on the set

$$\Delta^* = \{(y_1, y_2)\colon 0 \le y_2 \le y_1\},$$

but the systems (Δ, T) and (Δ^*, T^*) are not differentiably isomorphic in the sense of Eq. (1). To say it more precisely, no fractional linear map ψ with this property exists.

However, (Δ^*, T^*) is an exceptional dual as defined in SCHWEIGER [2]. The map

$$\psi(s, t) = \left(\frac{1}{s + t}, \frac{t}{s + t}\right)$$

has the property

$$\alpha(k) \circ \psi = \psi \circ \alpha^*(k)$$

for all $k = \varepsilon, (02), (012), (021)$ but

$$\begin{aligned}
\alpha(01) \circ \psi &= \psi \circ \alpha^*(12),\\
\alpha(12) \circ \psi &= \psi \circ \alpha^*(01).
\end{aligned}$$

This is reasonable since a closer inspection shows that the two maps $\alpha^*(\varepsilon)$ and $\alpha^*(12)$ lead to the same type of exceptional set as $\alpha(\varepsilon)$ and $\alpha(01)$ do for the map T.

References

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