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A Note on the Eigenvalues for Periodic Three-Dimensional Jacobi-Perron Algorithms

By

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Abstract

In their profound study on the connections between Lyapunov theory and approximation properties of Jacobi-Perron algorithm BROISE-ALAMICHEL and GUIVARC'H 2001 proved a generalization of an inequality due to PALEY and URSELL [2]. In this note this inequality is slightly refined for dimension $n = 3$. This shows that for the eigenvalues $\sigma_0 > |\sigma_1| \geq |\sigma_2| \geq |\sigma_3|$ of a periodic expansion the inequality $|\sigma_1\sigma_2| < 1$ is true. Furthermore it allows a more direct proof for the inequality $\lambda_1 + \lambda_2 < 0$ where $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_3$ are the Lyapunov exponents of the algorithm.

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1. Introduction

Let

$$T(x_1, x_2, x_3) = \left(\frac{x_2}{x_1} - a, \frac{x_3}{x_1} - b, \frac{1}{x_1} - c\right), \qquad a(x) = \left[\frac{x_2}{x_1}\right],$$
$$b(x) = \left[\frac{x_3}{x_1}\right], \qquad c(x) = \left[\frac{1}{x_1}\right]$$

denote the three-dimensional map related to Jacobi-Perron algorithm (PERRON [3], SCHWEIGER [4, 6]). Define

$$(a_s, b_s, c_s) = (a(T^{s-1}x), b(T^{s-1}x), c(T^{s-1}x)), \qquad 1 \le s.$$

Note that the digits satisfy the so-called *Perron conditions*

$$0 \le a_s \le c_s, \qquad 0 \le b_s \le c_s, \qquad 1 \le c_s.$$

Furthermore: If $a_s = c_s$ then $1 \le b_{s+1}$. If $b_s = c_s$ then $a_{s+1} \le b_{s+1}$. If $b_s = c_s$ and $a_{s+1} = b_{s+1}$ then $1 \le a_{s+2}$.

We introduce the matrices

$$\beta^{(s)} := \begin{pmatrix} c_s & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ a_s & 1 & 0 & 0 \\ b_s & 0 & 1 & 0 \end{pmatrix}$$

and

$$\beta^{(1)} \cdots \beta^{(s)} := \begin{pmatrix} A_0^{(s+4)} & A_0^{(s+1)} & A_0^{(s+2)} & A_0^{(s+3)} \\ A_1^{(s+4)} & A_1^{(s+1)} & A_1^{(s+2)} & A_1^{(s+3)} \\ A_2^{(s+4)} & A_2^{(s+1)} & A_2^{(s+2)} & A_2^{(s+3)} \\ A_3^{(s+4)} & A_3^{(s+1)} & A_3^{(s+2)} & A_3^{(s+3)} \end{pmatrix}.$$

From this notation we read the recursion relation

$$A_\alpha^{(s+4)} = c_s A_\alpha^{(s+3)} + b_s A_\alpha^{(s+2)} + a_s A_\alpha^{(s+1)} + A_\alpha^{(s)}, \qquad \alpha \in \{0, 1, 2, 3\}.$$

The following expansion is worth to be stated as a separate lemma.

Lemma (BROISE-ALAMICHEL and GUIVARC'H [1]).

$$\begin{aligned} A_0^{(s+4)} &= (c_s - 1)A_0^{(s+3)} + (c_{s-1} + b_s - 1)A_0^{(s+2)} \\ &\quad + (c_{s-2} + b_{s-1} + a_s - 1)A_0^{(s+1)} + (c_{s-3} + b_{s-2} + a_{s-1} + 1)A_0^{(s)} \\ &\quad + (b_{s-3} + a_{s-2} + 1)A_0^{(s-1)} + (a_{s-3} + 1)A_0^{(s-2)} + A_0^{(s-3)}. \end{aligned}$$

We denote by

$$[s+i, s+j, s+k]_{\alpha,\beta} = \det \begin{pmatrix} A_0^{(s+i)} & A_0^{(s+j)} & A_0^{(s+k)} \\ A_\alpha^{(s+i)} & A_\alpha^{(s+j)} & A_\alpha^{(s+k)} \\ A_\beta^{(s+i)} & A_\beta^{(s+j)} & A_\beta^{(s+k)} \end{pmatrix}$$

the relevant determinants. Since the choice of $\alpha, \beta \in \{1, 2, 3\}$ is not important we drop these indices.

The following recursion relations are valid,

$$[s+4,s+1,s+2] = c_s[s+3,s+1,s+2] + [s,s+1,s+2],$$
$$[s+4,s+1,s+3] = b_s[s+2,s+1,s+3] + [s,s+1,s+3],$$
$$[s+4,s+2,s+3] = a_s[s+1,s+2,s+3] + [s,s+2,s+3].$$

These relations lead to the following useful expansion,

$$[s+4,s+2,s+3] = a_s[s+1,s+2,s+3] + b_{s-1}[s+1,s,s+2] + c_{s-2}[s+1,s-1,s] + [s-2,s-1,s].$$

We introduce the quantity

$$\rho_s := \frac{|[s,s-1,s-2]|}{A_0^{(s)}}, \qquad s \geq 4.$$

Theorem.

$$|[s+4,s+2,s+3]| \leq A_0^{(s+4)}\left(1 - \frac{A_0^{(s-2)}}{A_0^{(s+4)}}\right) \max_{s-2\leq t\leq s+3} \rho_t.$$

Proof. The proof will be given by induction and by considering several cases. The relation

$$[s+4,s+2,s+3] = a_s[s+1,s+2,s+3] + b_{s-1}[s+1,s,s+2] + c_{s-2}[s+1,s-1,s] + [s-2,s-1,s]$$

immediately gives

$$|[s+4,s+2,s+3]| \leq (a_sA_0^{(s+3)} + b_{s-1}A_0^{(s+2)} + c_{s-2}A_0^{(s+1)} + A_0^{(s)}) \max_{s\leq t\leq s+3} \rho_t.$$

(1) If the following three conditions are satisfied, namely

$$a_s \leq c_s - 1, \qquad b_{s-1} \leq c_{s-1} + b_s - 1, \qquad c_{s-2} \leq c_{s-2} + b_{s-1} + a_s - 1,$$

then a comparison with

$$A_0^{(s+4)} = (c_s-1)A_0^{(s+3)} + (c_{s-1}+b_s-1)A_0^{(s+2)} + (c_{s-2}+b_{s-1}+a_s-1)A_0^{(s+1)} + (c_{s-3}+b_{s-2}+a_{s-1}+1)A_0^{(s)} + (b_{s-3}+a_{s-2}+1)A_0^{(s-1)} + (a_{s-3}+1)A_0^{(s-2)} + A_0^{(s-3)}$$

shows that

$$|[s+4,s+2,s+3]| \leq A_0^{(s+4)}\left(1 - \frac{A_0^{(s)}}{A_0^{(s+4)}}\right) \max_{s\leq t\leq s+3} \rho_t.$$

(2) Now assume that $c_s - 1 < a_s$. Then by Perron conditions we have $a_s = c_s$. Then we calculate

$$\begin{aligned}[s+4, s+2, s+3] &= c_s[s+1, s+2, s+3] + [s, s+2, s+3] \\ &= (c_s - 1)[s+1, s+2, s+3] \\ &\quad + [s+1, s+2, s+3] + [s, s+2, s+3] \\ &= (c_s - 1)[s+1, s+2, s+3] \\ &\quad + (a_{s-1} - b_{s-1})[s, s+1, s+2] \\ &\quad + (b_{s-2} - c_{s-2})[s, s-1, s+1] \\ &\quad + (1 + c_{s-3})[s, s-2, s-1] + [s-3, s-2, s-1].\end{aligned}$$

This gives the estimate

$$\begin{aligned}|[s+4, s+2, s+3]| \le \max_{s-1 \le t \le s+3} \rho_t((c_s - 1)A_0^{(s+3)} + |a_{s-1} - b_{s-1}|A_0^{(s+2)} \\ + |b_{s-2} - c_{s-2}|A_0^{(s+1)} + (1 + c_{s-3})A_0^{(s)} + A_0^{(s-1)}).\end{aligned}$$

We will show that the inequalities

$$|a_{s-1} - b_{s-1}| \le c_{s-1} + b_s - 1, \quad |b_{s-2} - c_{s-2}| \le c_{s-2} + b_{s-1} + a_s - 1$$

are satisfied. Then a comparison with the expansion in the lemma shows

$$|[s+4, s+2, s+3]| \le A_0^{(s+4)} \left(1 - \frac{A_0^{(s-2)}}{A_0^{(s+4)}}\right) \max_{s-2 \le t \le s+3} \rho_t.$$

Suppose that $a_{s-1} - b_{s-1} > c_{s-1} + b_s - 1$. Then $c_{s-1} + 1 \ge a_{s-1} + 1 > c_{s-1} + b_{s-1} + b_s$. Then $b_{s-1} = b_s = 0$ and $c_{s-1} = a_{s-1}$. But the last condition implies $1 \le b_s$, a contradiction.

If $b_{s-1} - a_{s-1} > c_{s-1} + b_s - 1$ then $c_{s-1} + 1 \ge b_{s-1} + 1 > c_{s-1} + a_{s-1} + b_s$. Hence $a_{s-1} = b_s = 0$ and $b_{s-1} = c_{s-1}$. But $b_{s-1} = c_{s-1}$ implies $a_s \le b_s$, hence $a_s = 0$, a contradiction.

If $c_{s-2} - b_{s-2} > c_{s-2} + b_{s-1} + a_s - 1$ then $1 > b_{s-2} + b_{s-1} + a_s$. This leads to $b_{s-2} = b_{s-1} = a_s = 0$ which again contradicts $a_s = c_s$.

(3) Therefore from now on we assume $a_s \le c_s - 1$. However, there are two conditions left which could be violated.

(3.1) We first assume $c_s \ge 2$.

(3.1.1) We consider the case

$$b_{s-1} > b_s + c_{s-1} - 1.$$

From $b_{s-1} \leq c_{s-1}$ we get $b_{s-1} = c_{s-1}$ and $b_s = 0$ and therefore also $a_s = 0$. Then we have the recursion

$$[s+4, s+2, s+3] = b_{s-1}[s+1, s, s+2] + c_{s-2}[s+1, s-1, s] + [s-2, s-1, s].$$

Then we estimate

$$|[s+4, s+2, s+3]| \leq (c_{s-1}A_0^{(s+2)} + c_{s-2}A_0^{(s+1)} + A_0^{(s)})\Big(\max_{s\leq t\leq s+2} \rho_t\Big).$$

This expression must be compared with

$$A_0^{(s+4)} \geq (c_s - 1)A_0^{(s+3)} + (c_{s-1} - 1)A_0^{(s+2)} + (c_{s-2} + c_{s-1} - 1)A_0^{(s+1)} + (c_{s-3} + 1)A_0^{(s)}.$$

Since

$$c_{s-1}A_0^{(s+2)} \leq A_0^{(s+3)} + (c_{s-1} - 1)A_0^{(s+2)}$$

we obtain

$$|[s+4, s+2, s+3]| \leq A_0^{(s+4)}\left(1 - \frac{A_0^{(s)}}{A_0^{(s+4)}}\right) \max_{s\leq t\leq s+3} \rho_t.$$

(3.1.2) Next suppose that

$$c_{s-2} > c_{s-2} + b_{s-1} + a_s - 1.$$

Then $a_s = b_{s-1} = 0$ and we find

$$[s+4, s+2, s+3] = c_{s-2}[s+1, s-1, s] + [s-2, s-1, s].$$

This leads to a comparison of

$$c_{s-2}A_0^{(s+1)} + A_0^{(s)}$$

with

$$A_0^{(s+4)} \geq (c_s - 1)A_0^{(s+3)} + (c_{s-1} + b_s - 1)A_0^{(s+2)} + (c_{s-2} - 1)A_0^{(s+1)} + A_0^{(s)}.$$

Since

$$c_{s-2}A_0^{(s+1)} \leq A_0^{(s+3)} + (c_{s-2} - 1)A_0^{(s+1)}$$

we get the same estimate as before.

(3.2) The remaining case is $c_s = 1$.

(3.2.1)

$$b_{s-1} > b_s + c_{s-1} - 1$$

leads to $b_{s-1} = c_{s-1}$ and $b_s = 0$ and $a_s = 0$. We expand the relation

$$[s+4, s+2, s+3] = b_{s-1}[s+1, s, s+2] + c_{s-2}[s+1, s-1, s] + [s-2, s-1, s]$$

to obtain

$$\begin{aligned}[s+4, s+2, s+3] = {} & (b_{s-1}-1)[s+1, s, s+2] \\ & + (c_{s-2}-a_{s-2})[s+1, s-1, s] \\ & - b_{s-3}[s-1, s-2, s] - c_{s-4}[s-1, s-3, s-2] \\ & + [s-2, s-1, s] - [s-4, s-3, s-2].\end{aligned}$$

As before we compare this relation with the expansion

$$\begin{aligned}A_0^{(s+4)} = {} & (c_s-1)A_0^{(s+3)} + (c_{s-1}+b_s-1)A_0^{(s+2)} \\ & + (c_{s-2}+c_{s-1}+a_s-1)A_0^{(s+1)} + (c_{s-3}+b_{s-2}+a_{s-1}+1)A_0^{(s)} \\ & + (b_{s-3}+a_{s-2}+1)A_0^{(s-1)} + (a_{s-3}+1)A_0^{(s-2)} + A_0^{(s-3)}.\end{aligned}$$

Obviously, $c_{s-2} - a_{s-2} \le c_{s-2} + c_{s-1} + a_s - 1$ is true. The remaining critical estimate is

$$(b_{s-3}+1)A_0^{(s)} + c_{s-4}A_0^{(s-1)} \le (c_{s-3}+b_{s-2}+a_{s-1}+1)A_0^{(s)} + (b_{s-3}+a_{s-2}+1)A_0^{(s-1)}$$

or equivalently

$$b_{s-3}A_0^{(s)} + c_{s-4}A_0^{(s-1)} \le (c_{s-3}+b_{s-2}+a_{s-1})A_0^{(s)} + (b_{s-3}+a_{s-2})A_0^{(s-1)}.$$

If $b_{s-3} = c_{s-3}$ then we obtain $a_{s-2} \le b_{s-2}$. If $b_{s-2} = 0$ then $a_{s-2} = 0$ and hence $a_{s-1} \ge 1$. The other case is $b_{s-1} \ge 1$. In both cases we obtain

$$c_{s-4}A_0^{(s-1)} \le A_0^{(s)} + A_0^{(s-1)}.$$

Hence

$$|[s+4, s+2, s+3]| \le A_0^{(s+4)}\left(1 - \frac{A_0^{(s-1)}}{A_0^{(s+4)}}\right) \max_{s-1 \le t \le s+3} \rho_t.$$

If $b_{s-3} < c_{s-3}$ then we get again

$$c_{s-4}A_0^{(s-1)} \le A_0^{(s)} + A_0^{(s-1)}$$

and the same result.

(3.2.2) If

$$c_{s-2} > c_{s-2} + b_{s-1} + a_s - 1,$$

then $b_{s-1} = a_s = 0$. Note that in this case clearly $b_{s-1} \le b_s + c_{s-1} - 1$. We look at

$$[s+4, s+2, s+3] = c_{s-2}[s+1, s-1, s] + [s-2, s-1, s]$$

and compare this with

$$A_0^{(s+4)} \ge (c_{s-2} - 1)A_0^{(s+1)} + (c_{s-3} + 1)A_0^{(s)} + (b_{s-3} + 1)A_0^{(s-1)} + (a_{s-3} + 1)A_0^{(s-2)} + A_0^{(s-3)}$$

which is equivalent to

$$A_0^{(s+4)} \ge c_{s-2}A_0^{(s+1)} + A_0^{(s)} + A_0^{(s-1)} + A_0^{(s-2)}.$$

This leads to the estimate

$$|[s+4, s+2, s+3]| \le A_0^{(s+4)} \left(1 - \frac{A_0^{(s-1)}}{A_0^{(s+4)}}\right) \max_{s-2 \le t \le s+1} \rho_t.$$

For a periodic algorithm with eigenvalues $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ ordered in a way such that $\sigma_0 > |\sigma_1| \ge |\sigma_2| \ge |\sigma_3|$ the following corollary follows.

Corollary 1. $|\sigma_1 \sigma_2| < 1$.

Proof. Note that for a periodic algorithm with period length p the relations

$$A_0^{(sp+4)} + A_1^{(sp+1)} + A_2^{(sp+2)} + A_3^{(sp+3)} \sim \sigma_0^s$$

and

$$[sp+4, sp+1, sp+2]_{0,1,2} + [sp+1, sp+2, sp+3]_{1,2,3} + [sp+4, sp+2, sp+3]_{0,2,3} + [sp+4, sp+1, sp+3]_{0,1,3} \sim |\sigma_0 \sigma_1 \sigma_2|^s.$$

Here the quantities $[s+i, s+j, s+k]_{\alpha,\beta,\gamma}$ are defined as

$$[s+i, s+j, s+k]_{\alpha,\beta,\gamma} = \det \begin{pmatrix} A_\alpha^{(s+i)} & A_\alpha^{(s+j)} & A_\alpha^{(s+k)} \\ A_\beta^{(s+i)} & A_\beta^{(s+j)} & A_\beta^{(s+k)} \\ A_\gamma^{(s+i)} & A_\gamma^{(s+j)} & A_\gamma^{(s+k)} \end{pmatrix}.$$

Clearly the theorem of the paper extends to these numbers. Since the algorithm is periodic there is a constant $q<1$ such that

$$1-\frac{A_0^{(s-2)}}{A_0^{(s+4)}}\leq q<1.$$

If $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are the four Lyapunov exponents the following result can be proved by applying the ergodic theorem.

Corollary 2. $\lambda_0+\lambda_3>0$.

Proof. The proof closely follows SCHWEIGER [5] (see also SCHWEIGER [4]). We introduce the quantity

$$\tau_s := \max_{0\leq j\leq 5} \rho_{s+j}.$$

Then we find

$$\tau_{s+6}\leq \max_{0\leq j\leq 5}\left(1-\frac{A_0^{(s+j-2)}}{A_0^{(s+j+4)}}\right)\tau_s.$$

Using estimates like

$$1-\frac{A_0^{(s-2)}}{A_0^{(s+4)}}\leq 1-\frac{1}{4^6 c_s c_{s-1} c_{s-2} c_{s-3} c_{s-4} c_{s-5}},$$

we see that the product

$$\prod_{s=1}^{N}\left(1-\max_{0\leq j\leq 5}\left(1-\frac{A_0^{(s+j-2)}}{A_0^{(s+j+4)}}\right)\right)^{1/N}$$

can be estimated almost everywhere by using the ergodic theorem. More precisely, there is a constant $\kappa<1$ such that

$$\prod_{s=1}^{N}\left(1-\max_{0\leq j\leq 5}\left(1-\frac{A_0^{(s+j-2)}}{A_0^{(s+j+4)}}\right)\right)^{1/N}\leq\kappa$$

holds almost everywhere. Since almost everywhere

$$\lim_{s\to\infty}\frac{\log A_0^{(s)}}{s}=\lambda_0,$$

the proof can be easily completed.

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