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Remarks on Some Sequences of Binomial Sums

By

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Abstract

We give simple proofs for the recurrence relations of some sequences of binomial sums which have previously been obtained by other more complicated methods.

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1. Introduction

Modifying an idea of BRIETZKE [2] we give simple proofs for the recurrence relations of sequences of binomial sums of the form

$$a(n,m,k,z) = \sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2} \right\rfloor},$$

which have been obtained by other methods in [3].

In order to motivate the method we consider first the well-known special case

$$a(n,5,k,-1) = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n-5j+k}{2} \right\rfloor} = (-1)^k \sum_j t(n,k-5j),$$

with

$$t(n,k) = (-1)^k \binom{n}{\left\lfloor \frac{n+k}{2} \right\rfloor}.$$

We use the fact that $t(n,k) = -t(n-1,k-1) - t(n-1,k+1)$ with $t(0,0) = 1, t(0,1) = -1$ and $t(0,k) = 0$ for all other $k \in \mathbb{Z}$.

Define the operator K by $Kf(n,k) = f(n,k-1)$ and the operator N by $Nf(n,k) = f(n+1,k)$. Then

$$t(n) = Nt(n-1) = -(K+K^{-1})t(n-1) = (-1)^n (K+K^{-1})^n t(0).$$

Let $s(n,k)$ on $\mathbb{N} \times \mathbb{Z}$ be the function which satisfies the same recurrence with initial values $s(0,k) = [k=0]$. Then we have $t(0) = (1-K)s(0)$. Since K is a linear operator we also have $t(n) = (1-K)s(n)$.

Let $\mathcal{F}$ be the vector space of all functions on $\mathbb{N} \times \mathbb{Z}$ which are finite linear combinations of functions $K^j s$, $j \in \mathbb{Z}$. For $f \in \mathcal{F}$ we have $Nf = -(K+K^{-1})f$.

Let T be the linear operator on $\mathcal{F}$ defined by

$$\begin{aligned} Tf &= N^2 f - Nf - f = (K+K^{-1})^2 f + (K+K^{-1})f - f \\ &= (K^{-2} + K^{-1} + 1 + K + K^2)f. \end{aligned}$$

Then

$$\sum_{j \in \mathbb{Z}} K^{5j} T K^i s(0) = \sum_{j \in \mathbb{Z}} K^j s(0) = 1 \qquad \text{for all} \quad i \in \mathbb{Z}$$

since $KT = TK$.

Furthermore

$$\begin{aligned} \sum_{j \in \mathbb{Z}} K^{5j} T t(n) &= \sum_{j \in \mathbb{Z}} K^{5j} T (-1)^n (K+K^{-1})^n (1-K)s(0) \\ &= (-1)^n (K+K^{-1})^n (1-K) \sum_{j \in \mathbb{Z}} K^{5j} T s(0) = 0. \end{aligned}$$

Since

$$a(n,5,k,-1) = (-1)^k \sum_j t(n,k-5j)$$

is a finite sum for each k, the sequence $(a(n,5,k,-1))$ satisfies the recurrence

$$a(n+2,5,k,-1) - a(n+1,5,k,-1) - a(n,5,k,-1) = 0 \qquad \text{for} \quad n \geq 0.$$

Since the Fibonacci numbers F_n satisfy the same recurrence with initial values $F_0 = 0$ and $F_1 = 1$, we get the following results (cf. ANDREWS [1]):

Proposition 1. *For* $k \equiv 0, 1 \pmod{10}$ *the initial conditions are* $a(0,5,k) = a(1,5,k) = 1$ *and therefore* $a(n,5,k) = F_{n+1}$.

For $k \equiv 2, 9 \pmod{10}$ *we have* $a(0,5,k) = 0$, $a(1,5,k) = 1$ *and therefore* $a(n,5,k) = F_n$.

For $k \equiv 3, 8 \pmod{10}$ *we get* $a(0,5,k) = a(1,5,k) = 0$ *and therefore* $a(n,5,k) = 0$. *Furthermore* $a(n,5,k+5) = -a(n,5,k)$.

It is interesting to observe that this result has first been proved by SCHUR [6] in a strengthened version: Let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^{n-k+1})\cdots(1-q^n)}{(1-q)\cdots(1-q^k)}$$

be a q-binomial coefficient. Then the following polynomial version of the celebrated Rogers-Ramanujan identity

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{k(5k-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n+5k}{2} \right\rfloor \end{bmatrix}$$

holds, which for $q = 1$ reduces to

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1} = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+5j}{2} \right\rfloor}.$$

An elementary proof of this q-identity may be found in [5].

2. A Useful Method

After this example let us consider a more general case.

For $a, b \in \mathbb{R}$ let $s_{a,b}$ be the function on $\mathbb{N} \times \mathbb{Z}$ defined by $s_{a,b}(0,k) = [k=0]$ and the recurrence relation

$$s_{a,b}(n,k) = as_{a,b}(n-1,k-1) + bs_{a,b}(n-1,k) + as_{a,b}(n-1,k+1). \tag{1}$$

This can be written in the form

$$s_{a,b}(n) = (aK^{-1} + b + aK)s_{a,b}(n-1) = (aK^{-1} + b + aK)^n s_{a,b}(0).$$

Let $\mathcal{F}$ be the vector space of all functions on $\mathbb{N}$ which are finite linear combinations of functions $K^j s_{a,b}$, $j \in \mathbb{Z}$.

For any polynomial

$$p(x) = \sum_{i=0}^{m} a_i x^i$$

we denote by $p(N)$ the linear operator on $\mathcal{F}$ defined by

$$p(N)f(n) = \sum_{i=0}^{m} a_i f(n+i).$$

Then we have $p(N) = p(aK^{-1} + b + aK)$.

We are looking for an operator $p(N)$ with analogous properties as T had in the above example.

To this end we define a sequence of polynomials

$$p_n(x,a,b) = \sum_{k=0}^{n} p_{n,k}(a,b)x^k$$

by the recurrence

$$p_n(x,a,b) = (x-b)p_{n-1}(x,a,b) - a^2 p_{n-2}(x,a,b) \tag{2}$$

with initial values $p_0(x,a,b) = 1$ and $p_1(x,a,b) = x + a - b$.

Lemma 1. *For all $k \in \mathbb{Z}$ the following identity holds*

$$p_m(N,a,b)s_{a,b}(0,k) = \sum_{i=0}^{m} p_{m,i}(a,b)s_{a,b}(i,k) = a^m[|k| \le m]. \tag{3}$$

Proof. It suffices to show that on $\mathcal{F}$

$$p_m(N,a,b) = a^m \sum_{j=-m}^{m} K^j. \tag{4}$$

It is immediately verified that (4) is true for $m = 0$ and $m = 1$, since

$$(N + a - b) = (aK + a + aK^{-1}).$$

If (4) has already been shown for $m-1$ and $m-2$ we get

$$\begin{aligned} p_m(N,a,b) &= (N-b)p_{m-1}(N,a,b) - a^2 p_{m-2}(N,a,b) \\ &= a(K + K^{-1})a^{m-1} \sum_{j=-m+1}^{m-1} K^j - a^2 a^{m-2} \sum_{j=-m+2}^{m-2} K^j \\ &= a^m \sum_{j=-m}^{m} K^j. \end{aligned}$$

From (3) we get

$$\sum_{i=0}^{m} p_{m,i}(a,b) \sum_{j \in \mathbb{Z}} s_{a,b}(i, k-(2m+1)j) = a^m \tag{5}$$

for each $k \in \mathbb{Z}$.

Application. As an application we consider for each $m \in \mathbb{N}$ the sequence

$$\begin{aligned} a(n, 2m+1, k, -1) &= \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n-(2m+1)j+k}{2} \right\rfloor} \\ &= (-1)^k \sum_j t(n, k-(2m+1)j). \end{aligned}$$

As shown above we have $t = (1-K)s_{-1,0}$. Therefore by (5) we get

$$\sum_{i=0}^{m} p_{m,i}(-1,0) a(0, 2m+1, k, -1) = 0.$$

Formula (1) implies that $t(n)$ is a finite linear combination of functions $K^j t(0)$. Therefore we also get

$$\begin{aligned} &p_m(N, -1, 0) a(n, 2m+1, k, -1) \\ &\quad = \sum_{i=0}^{m} p_{m,i}(-1,0) a(n, 2m+1, k, -1) = 0. \end{aligned}$$

Now we look for an explicit expression for $p_n(x, -1, 0)$.

We know that it satisfies the recurrence

$$p_n(x,-1,0) = x p_{n-1}(x,-1,0) - p_{n-2}(x,-1,0)$$

with initial values $p_0(x,-1,0) = 1$ and $p_1(x,-1,0) = x-1$.

Recall that the Fibonacci polynomials

$$\begin{aligned} F_n(x,s) &= \sum_{k=0}^{n-1} \binom{n-1-k}{k} s^k x^{n-2k-1} \\ &= \frac{1}{\sqrt{x^2+4s}} \left(\left(\frac{x+\sqrt{x^2+4s}}{2} \right)^n - \left(\frac{x-\sqrt{x^2+4s}}{2} \right)^n \right) \end{aligned} \tag{6}$$

are characterized by the recurrence

$$F_n(x,s) = xF_{n-1}(x,s) + sF_{n-2}(x,s) \tag{7}$$

with initial conditions $F_0(x,s) = 0$ and $F_1(x,s) = 1$. Therefore

$$p_n(x,-1,0) = F_{n+1}(x,-1) - F_n(x,-1).$$

The first values of the polynomials $p_n(x,-1,0)$ are

$$1, x-1, x^2-x-1, x^3-x^2-2x+1, x^4-x^3-3x^2+2x+1, \ldots.$$

This gives

Theorem 1. *The sequence*

$$a(n,2m+1,k,-1) = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n-(2m+1)j+k}{2} \right\rfloor}$$

satisfies the recurrence relation of order m

$$(F_{m+1}(N,-1) - F_m(N,-1))a(n,2m+1,k,-1) = 0 \tag{8}$$

for each $k \in \mathbb{Z}$.

Remark. This theorem has been proved in [3] with a more complicated method. The recurrence (8) is not for all k the minimal recurrence, because e.g. $a(n,2m+1,m+1,-1) \equiv 0$. But it is so for $a(n,2m+1,0,-1)$, which has a simple combinatorial interpretation. It is the number of the set of all lattice paths in $\mathbb{R}^2$ which start at the origin, consist of $\lfloor \frac{n}{2} \rfloor$ northeast steps $(1,1)$ and $\lfloor \frac{n+1}{2} \rfloor$ southeast steps $(1,-1)$ and which are contained in the strip $-m-1 < y < m$ (cf. e.g. [4], [5]).

It is easy to see that the initial values of $a(n,2m+1,0,-1)$ are

$$a(j,2m+1,0,-1) = \binom{j}{\left\lfloor \frac{j}{2} \right\rfloor} \qquad \text{for} \quad 0 \le j < 2m.$$

As a special case of Theorem 1 we mention that $a(n,3,0,-1) = 1$. This means

$$\sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n-3j}{2} \right\rfloor} = 1 \qquad \text{for all} \quad n \in \mathbb{N}.$$

The generating function of the sequence $(a(n,2m+1,0,-1))_{n\geq 0}$ has the form

$$\sum_{n\geq 0} a(n,2m+1,0,-1)x^n = \frac{c_m(x)}{d_m(x)},$$

where

$$\begin{aligned} d_m(x) &= p_m\left(\frac{1}{x},-1,0\right)x^m = x^m\left(F_{m+1}\left(\frac{1}{x},-1\right) - F_{m+1}\left(\frac{1}{x},-1\right)\right) \\ &= F_{m+1}(1,-x^2) - xF_m(1,-x^2) \end{aligned}$$

and $c_m(x)$ is a polynomial of degree less than m.

The first values of $(c_m(x))_{m\geq 1}$ are

$$\begin{aligned} &c_1(x) = 1, \qquad c_2(x) = 1, \qquad c_3(x) = 1 - x^2, \\ &c_4(x) = 1 - 2x^2, \qquad c_5(x) = 1 - 3x^2 + x^4, \ldots . \end{aligned}$$

This suggests that for $m \geq 2$

$$c_m(x) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1-j}{j} x^{2j} = F_m(1,-x^2).$$

This can be proved in the following way: Both $d_m(x)$ and $F_m(1,-x^2)$ satisfy the same recurrence $h_m(x) = h_{m-1}(x) - x^2 h_{m-2}(x)$. This implies that for

$$a_{2m+1}(x) = \sum_{n\geq 0} a(n,2m+1,0,-1)x^n$$

we have

$$\begin{aligned} &d_m(x)a_{2m+1}(x) - d_{m-1}(x)a_{2m-1}(x) + x^2 d_{m-2}(x)a_{2m-3}(x) \\ &\quad = (d_m(x) - d_{m-1}(x) - x^2 d_{m-2}(x))a_{2m+1}(x) + d_{m-1}(x)(a_{2m+1}(x) \\ &\qquad - a_{2m-1}(x)) + x^2 d_{m-2}(x)(a_{2m+1}(x) - a_{2m-3}(x)). \end{aligned}$$

Since the coefficients of x^j for $0 \leq j \leq 2m-5$ of $a_{2m-3}(x)$ are the same as those of $a_{2m-1}(x)$ and $a_{2m+1}(x)$ we see that for $2m-4 \geq m-1$ the polynomial

$$d_m(x)a_{2m+1}(x) - d_{m-1}(x)a_{2m-1}(x) + x^2 d_{m-2}(x)a_{2m-3}(x)$$

which has degree $<m$ must identically vanish. This implies that

$$c_m(x) = d_m(x)a_{2m+1}(x) = F_m(1,-x^2).$$

Corollary 1. *For $m \geq 2$ the generating function for $a(n, 2m+1, 0, -1)$ is given by*

$$\sum_{n \geq 0} a(n, 2m+1, 0, -1)x^n = \frac{F_m(1, -x^2)}{F_{m+1}(1, -x^2) - xF_m(1, -x^2)}. \tag{9}$$

3. A Modification of the Above Method

In order to obtain an analogous result for the sequences $a(n, 2m, k, -1)$ we define a sequence of polynomials

$$q_n(x, a, b) = \sum_{k=0}^{n} q_{n,k}(a, b)x^k$$

by the same recurrence

$$q_n(x, a, b) = (x - b)q_{n-1}(x, a, b) - a^2 q_{n-2}(x, a, b), \tag{10}$$

but with initial values $q_0(x, a, b) = 2$ and $q_1(x, a, b) = x - b$.

Lemma 2. *For all $k \in \mathbb{Z}$ the following identity holds*

$$q_m(N, a, b)s_{a,b}(0, k) = \sum_{i=0}^{m} q_{m,i}(a, b)s_{a,b}(i, k) = a^m[|k| = m]. \tag{11}$$

Proof. It suffices to show that on $\mathcal{F}$

$$q_m(N, a, b) = a^m(K^m + K^{-m}). \tag{12}$$

(12) is true for $m = 0$ and $m = 1$ by inspection.

If it is already shown for $m - 1$ and $m - 2$ we get

$$q_m(N, a, b) = a(K + K^{-1})a^{m-1}(K^{m-1} + K^{-(m-1)}) - a^2 a^{m-2}(K^{m-2} + K^{-(m-2)}) = a^m(K^m + K^{-m}).$$

Application. As an application let

$$u(n, k) = \binom{n}{\left\lfloor \frac{n+k}{2} \right\rfloor}.$$

Then $u(n, k) = u(n-1, k-1) + u(n-1, k+1)$ and $u(0, k) = [k \in \{0, 1\}]$. Therefore

$$u(n, k) = s_{1,0}(n, k) + s_{1,0}(n, k-1) \qquad \text{or} \qquad u = (1 + K)s_{1,0}.$$

We have

$$\begin{aligned}
a(n,2m,k,-1) &= \sum_{j\in\mathbb{Z}}(-1)^j \binom{n}{\left\lfloor\frac{n-(2m)j+k}{2}\right\rfloor} \\
&= \sum_{j\in\mathbb{Z}}\left(\binom{n}{\left\lfloor\frac{n-(2m)2j+k}{2}\right\rfloor}\right. \\
&\quad \left. - \binom{n}{\left\lfloor\frac{n-(2m)(2j+1)+k}{2}\right\rfloor}\right) \\
&= \sum_{j\in\mathbb{Z}}(s_{1,0}(n,k-4mj)-s_{1,0}(n,k-2m-4mj)) \\
&\quad + \sum_{j\in\mathbb{Z}}(s_{1,0}(n,k-1-4mj)-s_{1,0}(n,k-1-2m-4mj)).
\end{aligned}$$

Here we get

$$q_m(N,1,0)\sum_{j\in\mathbb{Z}}(s_{1,0}(0,i-4mj)-s_{1,0}(0,i-2m-4mj))=0$$

for each i,

because for $i-4mj=m$ we get $i-4mj-2m=-m$ and the sums cancel and for $i-4mj=-m$ we get $i-4m(j-1)-2m=m$. For other values the sum vanishes.

In the same way as above we conclude that

$$q_m(N,1,0)\sum_{j\in\mathbb{Z}}(s_{1,0}(n,i-4mj)-s_{1,0}(n,i-2m-4mj))=0$$

too.

In order to give a concrete representation of $q_m(x,1,0)$ recall that the Lucas polynomials

$$\begin{aligned}
L_n(x,s) &= \sum_{k=0}^{n-1}\binom{n-k}{k}\frac{n}{n-k}s^k x^{n-2k} \\
&= \left(\frac{x+\sqrt{x^2+4s}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+4s}}{2}\right)^n \qquad (13)
\end{aligned}$$

are characterized by the recurrence

$$L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s) \qquad (14)$$

with initial conditions $L_0(x,s) = 2$ and $L_1(x,s) = x$. Therefore $q_n(x,1,0) = L_n(x,-1)$.

The first values of the sequence $(L_n(x,-1))_{n\geq 1}$ are

$$x, \quad x^2-2, \quad x^3-3x, \quad x^4-4x^2+2, \ldots .$$

Theorem 2. *For $m \geq 1$ the sequence*

$$a(n,2m,k,-1) = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n-(2m)j+k}{2} \right\rfloor}$$

satisfies the recurrence relation

$$L_m(N,-1)a(n,2m,k,-1) = 0. \tag{15}$$

Remark. It should be noted that $a(n,2m,0,-1)$ has the following combinatorial interpretation. It is the number of the set of all lattice paths in $\mathbb{R}^2$ which start at the origin, consist of $\lfloor \frac{n}{2} \rfloor$ northeast steps $(1,1)$ and $\lfloor \frac{n+1}{2} \rfloor$ southeast steps $(1,-1)$ and which are contained in the strip $-m<y<m$ (cf. e.g. [5]).

The generating function of the sequence $(a(n,2m,0,-1))_{n\geq 0}$ is given by

$$\sum_{n\geq 0} a(n,2m,0,-1)x^n = \frac{c_m(x)}{d_m(x)},$$

where

$$d_m(x) = q_m\left(\frac{1}{x},1,0\right)x^m = x^m L_m\left(\frac{1}{x},-1\right) = L_m(1,-x^2)$$

and $c_m(x)$ is a polynomial of degree less than m.

The first values of $(c_m(x))_{m\geq 1}$ are

$$c_1(x) = 1, \quad c_2(x) = 1+x, \quad c_3(x) = 1+x-x^2,$$
$$c_4(x) = 1+x-2x^2-x^3, \quad c_5(x) = 1+x-3x^2-2x^3+x^4, \ldots .$$

This implies as above that

$$c_m(x) = F_m(1,-x^2) + xF_{m-1}(1,-x^2).$$

Corollary 2. *For $m \geq 2$ the generating function for $a(n,2m,0,-1)$ is given by*

$$\sum_{n\geq 0} a(n,2m,0,-1)x^n = \frac{F_m(1,-x^2) + xF_{m-1}(1,-x^2)}{L_m(1,-x^2)}. \tag{16}$$

4. Further Applications

4a) The same method can be applied to the general sum

$$a(n,m,k,z) = \sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2} \right\rfloor} = \sum_{j\in\mathbb{Z}} z^{2j} \binom{n}{\left\lfloor \frac{n-2mj+k}{2} \right\rfloor}$$
$$+ \sum_{j\in\mathbb{Z}} z^{2j-1} \binom{n}{\left\lfloor \frac{n-2mj+k+m}{2} \right\rfloor}.$$

Here we get

$$L_m(N,-1)a(0,m,k,z) = L_m(N,-1)\sum_{j\in\mathbb{Z}} z^{2j}u(0,k-2mj)$$
$$+ L_m(N,-1)\sum_{j\in\mathbb{Z}} z^{2j-1}u(0,k+m-2mj).$$

In this case we have

$$L_m(N,-1)u(0,k-2mj) = \begin{cases} 1, & \text{if } k = 2mj-m, \\ 1, & \text{if } k = 2mj+m, \\ 0, & \text{else,} \end{cases}$$

or

$$L_m(N,-1)u(0,k-2mj) = u(0,k-m-2mj) + u(0,k+m-2mj).$$

This implies

$$L_m(N,-1)a(0,m,k,z) = \sum_{j\in\mathbb{Z}} z^{2j}(u(0,k-m-2mj) + u(0,k+m-2mj))$$
$$+ \sum_{j\in\mathbb{Z}} z^{2j-1}(u(0,k+2m-2mj) + u(0,k-2mj))$$
$$= \left(z+\frac{1}{z}\right)a(0,m,k,z).$$

Thus we get

$$\left(L_m(N,-1) - \left(z+\frac{1}{z}\right)\right)a(0,m,k,z) = 0.$$

Theorem 3. *The sequence*

$$a(n,m,k,z) = \sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2} \right\rfloor}$$

satisfies the recurrence relation

$$\left(L_m(N,-1)-\left(z+\frac{1}{z}\right)\right)a(n,m,k,z)=0. \tag{17}$$

Remark. It is easy to see that the initial values of $a(n,m,0,z)$ are

$$a(n,m,0,z)=\binom{j}{\left\lfloor \frac{j}{2}\right\rfloor} \qquad \text{for} \quad 0\le j<m-1,$$

$$a(m-1,m,0,z)=\binom{m-1}{\left\lfloor \frac{m-1}{2}\right\rfloor}+\frac{1}{z},$$

$$a(m,m,0,z)=\binom{m}{\left\lfloor \frac{m}{2}\right\rfloor}+\frac{1}{z}+z.$$

The generating function of the sequence $(a(n,m,0,z))$ for $m\ge 1$ has the form

$$\sum_{n\ge 0}a(n,m,0,z)x^n=\frac{c_m(x,z)}{d_m(x,z)}$$

with

$$d_m(x,z)=x^m\left(L_m\left(\frac{1}{x},-1\right)-\left(z+\frac{1}{z}\right)\right)=d_m(x)-x^m\left(z+\frac{1}{z}\right)$$

and

$$c_m(x,z)=\frac{x^{m-1}}{z}+F_m(1,-x^2)+xF_{m-1}(1,-x^2).$$

Since $d_m(x)=L_m(1,-x^2)$ and $F_m(1,-x^2)+xF_{m-1}(1,-x^2)$ satisfy the same recurrence $h_m(x)=h_{m-1}(x)-x^2h_{m-2}(x)$ we get

$$\begin{aligned}
&\left(d_m(x)-x^m\left(z+\frac{1}{z}\right)\right)a_m(x)-\left(d_{m-1}(x)-x^{m-1}\left(z+\frac{1}{z}\right)\right)a_{m-1}(x)\\
&\quad+x^2\left(d_{m-2}(x)-x^{m-2}\left(z+\frac{1}{z}\right)\right)a_{m-2}(x)\\
&=d_{m-1}(x)(a_m(x)-a_{m-1}(x))+x^2d_{m-2}(x)(a_m(x)-a_{m-2}(x))\\
&\quad-x^m\left(z+\frac{1}{z}\right)a_m(x)+x^{m-1}\left(z+\frac{1}{z}\right)a_{m-1}(x)-x^m\left(z+\frac{1}{z}\right)a_{m-2}(x).
\end{aligned}$$

Since $d_m(0) = 1$ it is easy to verify that for $m \geq 3$

$$d_{m-1}(x)(a_m(x) - a_{m-1}(x)) = -\frac{x^{m-2}}{z} - x^{m-1}z + x^m(\cdots)$$

and

$$x^2 d_{m-2}(x)(a_m(x) - a_{m-2}(x)) = -\frac{x^{m-1}}{z} + x^m(\cdots).$$

Therefore we get

$$\begin{aligned} &d_m(x,z)a_m(x) - d_{m-1}(x,z)a_{m-1}(x) + x^2 d_{m-2}(x,z)a_{m-2}(x) \\ &\quad = -\frac{x^{m-2}}{z} + x^m(\cdots). \end{aligned}$$

Now the left-hand side must be a polynomial of degree less than m. Therefore we have in fact

$$d_m(x,z)a_m(x) - d_{m-1}(x,z)a_{m-1}(x) + x^2 d_{m-2}(x,z)a_{m-2}(x) = -\frac{x^{m-2}}{z}.$$

Now $c_m(x,z)$ satisfies the same recurrence. Since the initial values coincide, we get

Corollary 3. *For $m \geq 2$ the generating function for $a(n,m,0,z)$ is given by*

$$\sum_{n\geq 0} a(n,m,0,z)x^n = \frac{(x^{m-1}/z) + F_m(1,-x^2) + xF_{m-1}(1,-x^2)}{L_m(1,-x^2) - x^m(z + (1/z))}. \quad (18)$$

Remark. In the same way we get

$$\begin{aligned} &\sum_{n\geq 0} a(n, 2m+1, m+1, z)x^n \\ &\quad = \frac{(1+z)x^m(F_{m+1}(1,-x^2) + xF_m(1,-x^2))}{L_{2m+1}(1,-x^2) - x^{2m+1}(z + (1/z))}. \end{aligned}$$

For $z = -1$ the right-hand side vanishes and therefore we get again $a(n, 2m+1, m+1, -1) = 0$.

4b) For the special case $z = 1$ also simpler recurrences can be found. It is easy to verify that

$$\left(x + \frac{1}{x} - 2\right) F_m\left(x + \frac{1}{x}, -1\right)(1+x) = \frac{1}{x^m} - \frac{1}{x^{m-1}} - x^m + x^{m+1}.$$

This implies as above

$$(N-2)F_m(N,-1)u(0) = (K^m - K^{m-1} - K^{-m} + K^{-m-1})s_{1,0}(0).$$

Therefore we get

$$(N-2)F_m(N,-1)\sum_j K^{2jm}u(0)$$
$$= \sum_j K^{2jm}(K^m - K^{m-1} - K^{-m} + K^{-m-1})s_{1,0}(0) = 0.$$

From this we conclude as above

Theorem 4. *The sequence*

$$a(n,2m,k,1) = \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n-(2m)j+k}{2} \right\rfloor}$$

satisfies the recurrence relation

$$(N-2)F_m(N,-1)a(n,2m,k,1) = 0. \tag{19}$$

Corollary 4. *For $m \geq 1$ the generating function for $a(n,2m,0,1)$ is given by*

$$\sum_{n\geq 0} a(n,2m,0,1)x^n = \frac{F_m(1,-x^2) - xF_{m-1}(1,-x^2)}{(1-2x)F_m(1,-x^2)}. \tag{20}$$

4c) It is again easy to verify that

$$\left(L_m\left(x+\frac{1}{x},-1\right) - L_{m-1}\left(x+\frac{1}{x},-1\right)\right)(1+x)$$
$$= \frac{1}{x^m} - \frac{1}{x^{m-2}} - x^{m-1} + x^{m+1}.$$

Therefore we get

$$(L_m(K+K^{-1},-1) - L_{m-1}(K+K^{-1},-1))\sum_j K^{(2m-1)j}u(0)$$
$$= \sum_j K^{(2m-1)j}(K^m - K^{m-2} - K^{-m+1} + K^{-m-1})s_{1,0}(0) = 0.$$

This implies

Theorem 5. *The sequence*

$$a(n,2m-1,k,1) = \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n-(2m-1)j+k}{2} \right\rfloor}$$

satisfies the recurrence relation

$$(L_m(N,-1) - L_{m-1}(N,-1))a(n, 2m-1, k, 1) = 0. \tag{21}$$

Corollary 5. *For $m \geq 2$ the generating function for $a(n, 2m-1, 0, 1)$ is given by*

$$\sum_{n \geq 0} a(n, 2m-1, 0, 1)x^n = \frac{L_{m-1}(1, -x^2)}{L_m(1, -x^2) - xL_{m-1}(1, -x^2)}. \tag{22}$$

Remark. For the special cases $z = \pm 1$ numerator and denumerator of the generating function

$$\frac{(x^{m-1}/z) + F_m(1, -x^2) + xF_{m-1}(1, -x^2)}{L_m(1, -x^2) - x^m(z + (1/z))}$$

have common divisors which can be cancelled.

This can be verified by using the following identities, which are easily deduced from the representations (6) and (13) (cf. e.g. [3]):

$$L_{2m}(x,-1) - 2 = (x^2 - 4)(F_m(x,-1))^2,$$

$$L_{2m-1}(x,-1) - 2 = \frac{(L_m(x,-1) - L_{m-1}(x,-1))^2}{x-2},$$

$$L_{2m}(x,-1) + 2 = (L_m(x,-1))^2,$$

$$L_{2m-1}(x,-1) + 2 = (x+2)(F_m(x,-1) - F_{m-1}(x,-1))^2.$$

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