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Thomas' Family of Thue Equations over Imaginary Quadratic Fields, II

By

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Abstract

We completely solve the family of relative Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu,$$

where the parameter t, the root of unity μ and the solutions x and y are integers in the same imaginary quadratic number field. This is achieved using the hypergeometric method for $|t| \geq 53$ and BAKER's method combined with a computer search using continued fractions for the remaining values of t.

Let F be an irreducible form of degree at least 3 with integral coefficients and m be a nonzero integer. Then the Diophantine equation

$$F(x, y) = m$$

is called a *Thue* equation in honor of THUE [10] who proved that it has only finitely many solutions over the integers. Algorithms for solving single Thue equations over $\mathbb{Z}$ have been developed, see BILU and HANROT [1].

Starting with THOMAS [9] in 1990, several families of parametrized Thue equations (of positive discriminant) have been solved, cf. the surveys [4, 3].

In the last years, a few parametrized families of relative Thue equations where the parameter and the solutions are elements of an imaginary quadratic number field have been studied by the authors [6], by ZIEGLER [11, 12], and by JADRIJEVIĆ and ZIEGLER [7].

In [6], the parametrized family of Thue equations

$$x^3 - (t-1)x^2y - (t+2)xy^2 - y^3 = \mu, \tag{1}$$

for $x, y \in \mathbb{Z}_{\mathbb{Q}(t)}$, an imaginary quadratic integer t, a root of unity μ in $\mathbb{Z}_{\mathbb{Q}(t)}$ has been studied. This is the family that THOMAS [9] and MIGNOTTE [8] solved completely in the rational integer case. In [6], all solutions for $|t| > 3.023 \cdot 10^9$ have been found using BAKER's method. Furthermore, all solutions for $\operatorname{Re} t = -1/2$ were claimed to be listed. However, the proof of [6, Theorem 3] is incorrect (more precisely, the argument for excluding the possibility $\Lambda = 0$ in [6, Section 7] is invalid) and some solutions are missing in [6, Table 2].

By combining the hypergeometric method due to THUE and SIEGEL (for values $|t| \geq 53$) and lower bounds for linear forms in logarithms ("BAKER's method") together with a computer search (using continued fraction expansions) for $|t| < 53$, the Diophantine equation (1) can be solved *completely* for *all values of t*.

The details are discussed in [2]. The purpose of this note is to announce the corrected and complete result:

Theorem. *Let t be an integer in an imaginary quadratic number field, $t \notin \{(-1 \pm 3\sqrt{-3})/2\}$, $\mathbb{Z}_{\mathbb{Q}(t)}$ be the ring of integers of $\mathbb{Q}(t)$,*

$$F_t(X,Y) = X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 \in \mathbb{Z}_{\mathbb{Q}(t)}[X,Y],$$

and μ be a root of unity in $\mathbb{Q}(t)$.

Then all solutions $(x,y) \in \mathbb{Z}^2_{\mathbb{Q}(t)}$ to

$$F_t(x,y) = \mu \tag{2}$$

Table 1. Solutions (if contained in $\mathbb{Q}(t)$) to (2) for all t, where $\omega_3 = (1+\sqrt{-3})/2$

x	y	μ	x	y	μ	x	y	μ
0	1	-1	i	0	$-i$	$-1+\omega_3$	$1-\omega_3$	-1
-1	0	-1	0	i	i	ω_3	0	-1
1	-1	-1	$-i$	0	i	0	$1-\omega_3$	1
0	-1	1	i	$-i$	i	0	ω_3	1
-1	1	1	0	$-\omega_3$	-1	$-\omega_3$	0	1
1	0	1	0	$-1+\omega_3$	-1	$1-\omega_3$	$-1+\omega_3$	1
0	$-i$	$-i$	$-\omega_3$	ω_3	-1	$-1+\omega_3$	0	1
$-i$	i	$-i$	$1-\omega_3$	0	-1	ω_3	$-\omega_3$	1

Table 2. Overview on sporadic solutions to (2) for specific t

t	Number of solutions	$\max\{\lvert x\rvert^2, \lvert y\rvert^2\}$
-4	6	81
-2	6	9
-1	12	81
0	12	81
1	6	9
3	6	81
$-1 \pm 2i$	24	5
$-1 \pm 3i$	24	5
$\pm 2i$	24	5
$\pm 3i$	24	5
$-1 \pm \sqrt{-2}$	6	9
$-1 \pm 2\sqrt{-2}$	6	3
$\pm \sqrt{-2}$	6	9
$\pm 2\sqrt{-2}$	6	3
$-2 \pm 2\sqrt{-3}$	12	688
$(-3 \pm 3\sqrt{-3})/2$	24	7
$-1 \pm \sqrt{-3}$	24	3
$-1 \pm 2\sqrt{-3}$	6	1
$(-1 \pm \sqrt{-3})/2$	18	27
$\pm \sqrt{-3}$	24	3
$\pm 2\sqrt{-3}$	6	1
$(1 \pm 3\sqrt{-3})/2$	24	7
$1 \pm 2\sqrt{-3}$	12	688
$-2 \pm \sqrt{-5}$	6	86
$1 \pm \sqrt{-5}$	6	86
$-1 \pm \sqrt{-7}$	12	4
$(-1 \pm \sqrt{-7})/2$	6	7
$\pm \sqrt{-7}$	12	4
$(-3 \pm \sqrt{-11})/2$	6	20
$(1 \pm \sqrt{-11})/2$	6	20
$(-1 \pm \sqrt{-19})/2$	6	19
$(-1 \pm \sqrt{-31})/2$	6	98
$(-1 \pm \sqrt{-35})/2$	6	611

are listed in Table 1 (*solutions independent of* t) *and in the online table* [5] (*solutions for specific values of* t). *A short summary of these* 732 "*sporadic*" *solutions is given in Table* 2. *The sporadic solutions with* $\operatorname{Re} t = -1/2$ *are listed in Table* 3.

Remark. If $t \in \{(-1 \pm 3\sqrt{-3})/2\}$ then $F_t(X, Y)$ is the cube of a linear polynomial. Thus (2) has infinitely many solutions (x, y) for all roots of unity $\mu \in \mathbb{Q}(\sqrt{-3})$ in this case.

Table 3. Sporadic solutions to $F_t(x,y)=1$ for $\mathrm{Re}\,t=-1/2$. The solutions to $F_t(x,y)=-1$ are the negatives of the listed values. There are no solutions to $F_t(x,y)=\mu$ for roots of unity μ other than for $\mu\in\{-1,1\}$ for $\mathrm{Re}\,t=-1/2$

t	x	y
$(-1\pm\sqrt{-3})/2$	$\pm 3\sqrt{-3}$	$(1\pm 3\sqrt{-3})/2$
$(-1\pm\sqrt{-3})/2$	$(-5\pm\sqrt{-3})/2$	$-2\pm\sqrt{-3}$
$(-1\pm\sqrt{-3})/2$	$(5\pm\sqrt{-3})/2$	$(-9\pm 3\sqrt{-3})/2$
$(-1\pm\sqrt{-3})/2$	$-2\pm\sqrt{-3}$	$(9\pm 3\sqrt{-3})/2$
$(-1\pm\sqrt{-3})/2$	$2\pm\sqrt{-3}$	$(5\pm\sqrt{-3})/2$
$(-1\pm\sqrt{-3})/2$	$(-9\pm 3\sqrt{-3})/2$	$2\pm\sqrt{-3}$
$(-1\pm\sqrt{-3})/2$	$(-1\pm 3\sqrt{-3})/2$	$\pm 3\sqrt{-3}$
$(-1\pm\sqrt{-3})/2$	$(1\pm 3\sqrt{-3})/2$	$(-1\pm 3\sqrt{-3})/2$
$(-1\pm\sqrt{-3})/2$	$(9\pm 3\sqrt{-3})/2$	$(-5\pm\sqrt{-3})/2$
$(-1\pm\sqrt{-7})/2$	$\pm\sqrt{-7}$	$(-1\pm\sqrt{-7})/2$
$(-1\pm\sqrt{-7})/2$	$(-1\pm\sqrt{-7})/2$	$(1\pm\sqrt{-7})/2$
$(-1\pm\sqrt{-7})/2$	$(1\pm\sqrt{-7})/2$	$\pm\sqrt{-7}$
$(-1\pm\sqrt{-19})/2$	$\pm\sqrt{-19}$	$(-3\pm\sqrt{-19})/2$
$(-1\pm\sqrt{-19})/2$	$(-3\pm\sqrt{-19})/2$	$(3\pm\sqrt{-19})/2$
$(-1\pm\sqrt{-19})/2$	$(3\pm\sqrt{-19})/2$	$\pm\sqrt{-19}$
$(-1\pm\sqrt{-31})/2$	$\pm\sqrt{-31}$	$(-19\pm\sqrt{-31})/2$
$(-1\pm\sqrt{-31})/2$	$(-19\pm\sqrt{-31})/2$	$(19\pm\sqrt{-31})/2$
$(-1\pm\sqrt{-31})/2$	$(19\pm\sqrt{-31})/2$	$\pm\sqrt{-31}$
$(-1\pm\sqrt{-35})/2$	$\pm 2\sqrt{-35}$	$24\pm\sqrt{-35}$
$(-1\pm\sqrt{-35})/2$	$-24\pm\sqrt{-35}$	$\pm 2\sqrt{-35}$
$(-1\pm\sqrt{-35})/2$	$24\pm\sqrt{-35}$	$-24\pm\sqrt{-35}$

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