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CPTM Symmetry for the Dirac Equation and Its Extended Version Based on the Vector Representation of the Lorentz Group

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We revisit the CPT theorem for the Dirac equation and its extended version based on the vector representation of the Lorentz group. Then it is proposed that CPTM may apply to this fundamental equation for a massive fermion a s a singlet or a doublet with isospin. The symbol M stands here for reversing the sign of the mass in the Dirac equation, which can be accomplished by operation on it with the so-called gamma-five matrix that plays an essential role for the chirality in the Standard Model. We define the CPTM symmetry for the standard and extended Dirac equation and discuss its physical implications and some possible consequences for general relativity.

Keywords: CPT theorem, Dirac equation, chirality, mass sign reversal, beyond standard model

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1. INTRODUCTION

The famous CPT invariance of quantum field theory means that the reversal of the sign of the charge (charge conjugation, C), and of the space-time coordinates (parity reflection P and time reversal T) are fundamental physical symmetries [1–6] of any field equations, and the related Lagrangians thus should be invariant under these transformations. Lehnert [7] provides a concise modern review of CPT symmetry and its proofs and possible violation.

It seems that the combined CPT symmetry cannot be easily violated, but it has been put into question, for example recently by Heald [8]. In his conclusion "*CPT symmetry has been violated in relation to beta decay of mesons and for the three inverted CPT operations of antimatter that result in odd parity, and so it cannot be a fundamental symmetry of Nature. In the case of the broken CPT symmetry of Kaon beta decay for instance, the additional hidden symmetry can be provided by the negative mass of antimatter contained within the meson.*"

Checking the equality of the masses of particles and antiparticles is the adequate empirical test of the validity of CPT invariance within the Standard Model (SM). In a recent letter, Larin [9] also argued that there is a possibility of violation of CPT symmetry in the SM, yet consistent with the CPT theorem, by choosing non-standard phases of the quark fields. To check this experimentally requires by an order of magnitude higher precision of measurements of the proton and anti-proton mass difference than possible presently. Some recent experimental achievements in this field are discussed by Ulmer et al. [10]. Barenboim and Salvado [11] recently put forward ideas and made related calculations indicating that cosmology provides nowadays the most sensitive data to test CPT violation by scrutinizing the possible neutrino-antineutrino mass splitting, and by thus providing a bound on CPT violation for the absolute value of that splitting from cosmological data.

The main physical motivation of this paper is to show that there is another new symmetry for the Dirac equation, namely mass inversion M. The symmetries of the Dirac equation are not complete with the combined CPT symmetry, but due to the chirality (associated with the γ_5 matrix) there is another symmetry, namely mass inversion. However, by allowing the mass sign inversion to extend the CPT transformation into CPTM, the maximal symmetry of the Dirac equation based on the five gamma matrices is achieved. CPTM transformation is then simply proportional to the unity operator in Dirac-matrix terms. The fact that CPTM is proportional to the unit matrix means that the Dirac spinor field obeys it in all its spinor components like a scalar field, which it should as the squared Dirac equation reproduces the Klein-Gordon equation, as will be shown in Equations (23) and (28). This has no basic implications for a free fermion spinor field, yet may so in the context of coupling to gauge fields or gravity.

We stress at the outset that this paper is essentially a theoretical work which discusses CPTM as an extension of the symmetry of the Dirac equation in Minkowski space. According to our knowledge this idea in the present algebraic form has first been discussed by Marsch and Narita [12] before. We further define the CPTM symmetry for the new extended Dirac equation as proposed by Marsch and Narita [13, 14] and provide here the related detailed algebra. There it was also shown that the extended Dirac equation can easily be expanded to include the $SU(N)$ gauge fields of the standard model.

Especially, in the context of modern cosmology and general relativity, an extension of the CPT theorem has been considered by Bondarenko [15], which addresses the possibility of a negative sign of the particle mass, and related with it the mass sign-inversion symmetry. Therefore, in order to preserve fundamental invariance, the CPTM theorem has been proposed by him, whereby here the symbol M stands for the inversion of the sign of the mass m. CPTM was so defined to include besides charge-conjugation, parity reflection, and time-reversal now also the sign-inversion of the mass. As a result, antimatter might attain a negative "gravitational mass," and so CPTM invariance is required and must be ensured by that generalized theory.

But observationally, the gravitational behavior of antimatter is not known, and the interaction between matter and antimatter might be either attractive or repulsive. Villata [16] theoretically investigated a decade ago the behavior under CPT of the covariant equations of motion of an electron in a gravitational and electromagnetic field and came to the conclusion that there should be gravitational repulsion between matter and antimatter.

Bondarenko [15] originally had in mind the possible application of the CPTM symmetry of the space-time solutions of Einstein equations to the resolution of the cosmological constant problem. He also discussed the consequences of the CPTM symmetry for cosmological problems such as the values of the vacuum energy density and cosmological constant. But he further considered possible consequences of CPTM symmetry when being applied to quantum field theory and relativity.

Here we aim, within a much narrower scope, at revisiting the CPT theorem for the extended and original Dirac equation [17] and are led to propose that CPTM may, by good physical reasons, apply as well to this fundamental equation for a massive fermion described within the framework of special relativity. Changing the sign of the mass in the Dirac equation can be accomplished by operation of the so-called gamma-five matrix that plays an essential role for chirality in the SM. After revisiting the CPT theorem for the extended Dirac equation based on the vector representation of the Lorentz group [13], we newly define CPTM and discuss its consequences. A brief final section addresses some related theoretical aspects of general relativity. In a short **Appendix** we present some relevant matrices.

2. REVISITING THE CPT THEOREM

In this section we derive and discuss the basic symmetries of the standard and extended (including isospin) Dirac equation, which are charge conjugation, parity reflection and time inversion. We write the Dirac equation by using capital (lower) letters for the Gamma (gamma) matrices corresponding to the extended [13] and (standard) [17] Dirac equation. Lack of space forbids to explicitly quote here all related matrices in the Weyl representation. Some of them are quoted below in the **Appendix**. For more on this we refer to the cited references. The extended Dirac equation also has the usual form

$$\Gamma^{\mu}\mathrm{i}\partial_{\mu}\Phi = m\Phi. \tag{1}$$

A Dirac spinor field is denoted conventionally as ψ, whereas for the extended Dirac equation we use throughout the capital symbol Φ for the quantum field. To include an electromagnetic gauge field A_μ the derivative ∂_μ in Equation (1) is replaced by the covariant derivate

$$D_{\mu} = \partial_{\mu} + \mathrm{i}eA_{\mu}. \tag{2}$$

The three basic symmetries consist of the simple operations of complex conjugation, named C, parity reflection or space inversion, named P, and time inversion, named T. They have the following effects on the field: $\mathsf{C}\Phi = \Phi^*$, $\mathsf{P}\Phi(\mathbf{x}) = \Phi(-\mathbf{x})$, and $\mathsf{T}\Phi(t) = \Phi(-t)$. These operations are their own inversions, i.e., we have $\mathsf{C} = \mathsf{C}^{-1}$, $\mathsf{P} = \mathsf{P}^{-1}$, and $\mathsf{T} = \mathsf{T}^{-1}$. Furthermore, there are also three relevant matrices involved, such that we can generally define the symmetry operations as

$$\begin{aligned} \mathcal{C} &= C\,\mathsf{C}, \\ \mathcal{P} &= P\,\mathsf{P}, . \\ \mathcal{T} &= T\,\mathsf{T} \end{aligned} \tag{3}$$

These matrices (C, P, T) have like the Dirac matrices the dimension 8×8 and still need to be determined by explicit operation of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ on the Dirac equation. The transformed fields are identified by a calligraphic subscript. We define $\Phi_{\mathcal{C}} = C\mathsf{C}\Phi$, $\Phi_{\mathcal{P}} = P\mathsf{P}\Phi$, and $\Phi_{\mathcal{T}} = T\mathsf{T}\Phi$. Correspondingly, we obtain the three Dirac equations for the transformed fields reading

$$\begin{aligned} \Gamma^{\mu}\mathrm{i}D^{*}_{\mu}\Phi_{\mathcal{C}} &= m\Phi_{\mathcal{C}}, \\ \Gamma^{\mu}\mathrm{i}\partial_{\mu}\Phi_{\mathcal{P}} &= m\Phi_{\mathcal{P}},. \\ -(\Gamma^{\mu})^{*}\mathrm{i}\partial_{\mu}\Phi_{\mathcal{T}} &= m\Phi_{\mathcal{T}}. \end{aligned} \tag{4}$$

For the time inversion we had to first complex conjugate the kinetic term, as the Hamiltonian corresponding to the Dirac Lagrangian is real, and thus its eigenvalues must be real as well (see for example the related discussions in Refs. [4–6]). The above kinetic terms define the requirements to be posed on the matrices C, P, T such that transformed fields as well obey the original Dirac equation [Equation (1)]. Insertion of the operations given in Equation (3) and exploiting the definitions of $\mathsf{C}, \mathsf{P}, \mathsf{T}$ yields

$$\begin{aligned} (C^{-1}\Gamma^{\mu}C)\mathrm{i}D^{*}_{\mu}\Phi^{*} &= m\Phi^{*}, \\ (P^{-1}\Gamma^{\mu}P)\mathrm{i}\partial_{\mu}\Phi(-\mathbf{x}) &= m\Phi(-\mathbf{x}), \\ -(T^{-1}(\Gamma^{\mu})^{*}T)\mathrm{i}\partial_{\mu}\Phi(-t) &= m\Phi(-t). \end{aligned} \quad . \tag{5}$$

Concerning charge conjugation, we compare the first of the above three equations with the complex conjugated Dirac equation reading

$$(\Gamma^{\mu})^{*}(-\mathrm{i})D^{*}_{\mu}\Phi^{*} = m\Phi^{*}, \tag{6}$$

from which we read off by comparison that

$$C^{-1}\Gamma^{\mu}C = -(\Gamma^{\mu})^{*}. \tag{7}$$

Similarly, by inversion of the spatial and temporal variables in the original Dirac equation [Equation (1)], we obtain

$$\begin{aligned} \Gamma^{\mu}\mathrm{i}\partial_{\mu}(-\mathbf{x},t)\Phi(-\mathbf{x},t) &= m\Phi(-\mathbf{x},t), \\ \Gamma^{\mu}\mathrm{i}\partial_{\mu}(\mathbf{x},-t)\Phi(\mathbf{x},-t) &= m\Phi(\mathbf{x},-t). \end{aligned} \tag{8}$$

Here the derivate is $\partial_{\mu}(\mathbf{x},t) = (\partial/\partial t, \partial/\partial\mathbf{x})$. Again by comparison of Equation (5) with Equation (8), we find that the relations must hold:

$$P^{-1}\boldsymbol{\Gamma}P = -\boldsymbol{\Gamma}, \quad \text{and} \quad P^{-1}\Gamma_0 P = \Gamma_0. \tag{9}$$

Finally, for time inversion we obtain

$$T^{-1}(\boldsymbol{\Gamma})^{*}T = -\boldsymbol{\Gamma}, \quad \text{and} \quad T^{-1}(\Gamma_0)^{*}T = \Gamma_0. \tag{10}$$

So the remaining task is to determine the three matrices C, P, T for the extended Dirac equation. For parity close inspection of Equation (9) readily shows that $P = \Gamma_0$, and therefore

$$\mathcal{P} = \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix} \mathsf{P}. \tag{11}$$

For time inversion we need to know $(\Gamma^{\mu})^{*}$, which is obtained by complex conjugation of Equation (A3) in the **Appendix**. We thus obtain $\Gamma_0^{*} = -\Gamma_0$ and

$$\boldsymbol{\Gamma}^{*} = \mathrm{i}\begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}^{+} \\ \Delta\boldsymbol{\Sigma}^{-} & 0 \end{pmatrix}. \tag{12}$$

For time inversion close inspection of Equation (10) then suggests that

$$T = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix}. \tag{13}$$

For charge conjugation again close inspection now of Equation (7) leads to the matrix

$$C = \begin{pmatrix} 0 & \mathrm{i}1_4 \\ -\mathrm{i}1_4 & 0 \end{pmatrix}. \tag{14}$$

As a result, we obtained the mathematical expressions for all three symmetry operations. Now we can state their combined effects in the famous CPT Theorem of the Standard Model. The result is

$$\mathcal{CPT} = \begin{pmatrix} 0 & \mathrm{i}1_4 \\ -\mathrm{i}1_4 & 0 \end{pmatrix} \mathsf{C} \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix} \mathsf{P} \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} \mathsf{T} = \Gamma_5 \mathsf{CPT}. \tag{15}$$

This operator transforms the field as follows

$$\mathcal{CPT}\Phi(\mathbf{x},t) = \Gamma_5\Phi^{*}(-\mathbf{x},-t). \tag{16}$$

This result is the analog to the well known result for the normal Dirac equation as used in the SM of elementary particle physics [4], which reads similarly

$$\mathcal{CPT}\psi(\mathbf{x},t) = \gamma_5\psi^{*}(-\mathbf{x},-t). \tag{17}$$

Here the three involved standard Dirac 4×4 matrices are

$$C = \begin{pmatrix} 0 & \mathrm{i}\sigma_{\mathrm{y}} \\ -\mathrm{i}\sigma_{\mathrm{y}} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \mathrm{i}\sigma_{\mathrm{y}} & 0 \\ 0 & \mathrm{i}\sigma_{\mathrm{y}} \end{pmatrix}. \tag{18}$$

The definition of C differs by a minus sign from the conventional definition. The product of the three CPT matrices yields the diagonal matrix $\gamma_5 = [-1,-1,1,1]$. The Dirac matrices in Weyl representation attain the standard form quoted in the **Appendix**. When one replaces in Equation (5) the field Φ by ψ and Γ^{μ} by γ^{μ} and inserts the matrices of Equation (18) and the standard Dirac gamma matrices, then one finds that the same requirements as given in Equations (7, 9, 10) hold for, and are fulfilled by, the matrix γ^{μ}. More generally, one may consider phase factors of module unity in each of the above symmetry operations, an issue that is discussed in detail by Lehnert [7], but we take here trivial values of unity which suffices for the present purpose and the case we want to make.

3. EXTENSION TO CPTM SYMMETRY INVOKING MASS SIGN-INVERSION

It is interesting to note that the conventional $\mathcal{CPT}$ operation involves the Γ_5 matrix for the extended and γ_5 for the standard Dirac equation. We find this result puzzling because each component of the spinors ψ and Φ obeys the quadratic scalar Klein-Gordon (KG) equation [18], which has only the complex conjugation and space-time inversion as basic symmetries, which means CPT, and does not involve any Dirac matrices. We suggest that a possible solution to this problem may be found in another intrinsic symmetry of the Dirac equation. Consequently, we define similarly to what we did before the mass sign-version symmetry as $\Phi_{\mathcal{M}} = \mathcal{M}\Phi = M\mathsf{M}\Phi$. The operation M has the following effect on the massive-particle field, $\mathsf{M}\Phi(m) = \Phi(-m)$, and is of course its own inversion, i.e., giving $\mathsf{M} = \mathsf{M}^{-1}$. We recall that the solution of the Dirac equation depends on the mass as characteristic parameter of the particle. Still, the matrix M needs to be defined. One now obtains the Dirac equation in the two equivalent forms

$$\begin{aligned} \Gamma^\mu \mathrm{i}\partial_\mu \Phi_{\mathcal{M}} &= m\Phi_{\mathcal{M}}, \\ \Gamma^\mu \mathrm{i}\partial_\mu \Phi(-m) &= -m\Phi(-m). \end{aligned} \quad (19)$$

By comparison, we then obtain by insertion of $\Phi_{\mathcal{M}}$ a condition on the matrix M that is given by the equation

$$M^{-1}\Gamma^\mu M = -\Gamma^\mu. \quad (20)$$

Consequently, $M = \Gamma_5$, since this delivers by anticommutation with the Gammas the negative sign in front of the mass ($m \geq 0$). We emphasize that with that definition we find after Equation (15) that $\mathcal{CPTM} = 1_8\mathtt{CPTM}$, which is an important result. Let us briefly return to the KG equation. Formally, on can of course even in that equation,

$$(\partial^\mu \partial_\mu + m^2)\Phi = 0, \quad (21)$$

replace m by $-m$, which is trivial but may be denoted as a "mute mass sign-inversion."

Consequently, could $\mathcal{M}$ perhaps be the sought after additional symmetry of the Dirac equation, which would also apply to neutral but massive neutrinos, for example? Taking the roots of m^2 of course delivers $\pm m$, where the negative sign is conventionally not considered. As the Dirac equation results from taking in Lorentz-invariant fashion the "roots" of the KG equation, there is may be some physical meaning to the negative mass, corresponding to a negative frequency in the particles rest frame. Let us first consider the operation of $\mathtt{CPTM}$ on simple waves. As is well-known, fermion quantum fields can be Fourier decomposed, i.e., be written as a sum of plane waves with negative (particles) and positive (antiparticles) frequency like

$$\phi_\pm(\mathbf{x}, t) = \exp\left(\pm \mathrm{i}(E(p)t - \mathbf{p}\cdot\mathbf{x})\right), \quad (22)$$

with the positive definite relativistic particle energy $E(p) = \sqrt{p^2 + m^2}$ and momentum $\mathbf{p}$. It can then readily be seen that

$$\mathtt{CPTM}\phi_\pm(m; \mathbf{x}, t) = \phi^*_\pm(-m; -\mathbf{x}, -t) = \phi_\pm(m; \mathbf{x}, t). \quad (23)$$

Therefore, the plane waves are eigenfunctions of CPTM. Moreover, the relativistic four-momentum $P_\mu = \mathrm{i}\partial_\mu$ obeys the equation $(\mathtt{CPTM})^{-1}P_\mu\mathtt{CPTM} = P_\mu$, which means it commutes with the CPTM operation, since $\mathtt{M}$ has no effect on the particle energy depending on the square of the mass m.

To substantiate our suggestion we present a few more instructive calculations for the individual symmetry operations. For the sake of convenience we provide them only for the standard Dirac equation. For the extended one we simply have to replace the lower case with the upper gamma matrices, and similarly for C, P, T, and M as defined above. By insertion it is easily validated that the particle and antiparticle solutions of the free Dirac equation can be written

$$\begin{aligned} \psi_{\mathrm{P}}(\mathbf{p}, m; \mathbf{x}, t) &= (\gamma\, p + m)u \exp(-\mathrm{i}p\, x), \\ \psi_{\mathrm{A}}(\mathbf{p}, m; \mathbf{x}, t) &= (\gamma\, p - m)v \exp(+\mathrm{i}p\, x). \end{aligned} \quad (24)$$

Here we use the abbreviation $\gamma\, p = \gamma^\mu p_\mu$, with $p^\mu = (E(p), \mathbf{p})$. Similarly, we have $p\, x = p^\mu x_\mu$, with $x^\mu = (t, \mathbf{x})$. The vectors u and v form a basis in the fermion rest frame and are orthogonal and can be normalized in a Lorentz-invariant fashion. Explicitly they read in Weyl representation as follows: $u_1^T = (1, 0, 1, 0)$, $u_2^T = (0, 1, 0, 1)$, $v_1^T = (1, 0, -1, 0)$, $v_2^T = (0, 1, 0, -1)$, whereby index 1 and 2 refers to spin up and down. When we operate with the charge conjugation operator on Equation (24) and use that $C = C^*$ and $C^2 = 1_4$, we find that

$$\begin{aligned} \mathcal{C}\psi_{\mathrm{P}1,2} &= C(\gamma^*\, p + m)u^*_{1,2} \exp(+\mathrm{i}p\, x) \\ &= (C\gamma^* C\, p + m)Cu^*_{1,2} \exp(+\mathrm{i}p\, x) = \pm\psi_{\mathrm{A}2,1}, \\ \mathcal{C}\psi_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= \pm\psi_{\mathrm{A}2,1}(\mathbf{p}, m; \mathbf{x}, t), \end{aligned} \quad (25)$$

since $Cu^*_{1,2} = \mp v_{2,1}$, whereby the spin is flipped. As expected, the particle is transformed into the antiparticle solution with opposite spin, and vice versa, which is not shown here. Exploiting the relations Equations (9) and (10) with Gamma being replaced by gamma, and using the definitions given in Equation (18), one finds after some algebra that

$$\begin{aligned} \mathcal{P}\psi_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= +\psi_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t), \\ \mathcal{P}\psi_{\mathrm{A}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= -\psi_{\mathrm{A}1,2}(\mathbf{p}, m; \mathbf{x}, t), \\ \mathcal{T}\psi^*_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= \mp\psi_{\mathrm{P}2,1}(\mathbf{p}, m; \mathbf{x}, t), \\ \mathcal{T}\psi^*_{\mathrm{A}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= \mp\psi_{\mathrm{A}2,1}(\mathbf{p}, m; \mathbf{x}, t). \end{aligned} \quad (26)$$

Here we used the fact that in the rest frame the basis vectors behave as follows against time and space inversion, i.e., we have $Pu_{1,2} = u_{1,2}$, and $Pv_{1,2} = -v_{1,2}$, which means that particles have positive and antiparticles negative rest-frame parity, and moreover $Tu^*_{1,2} = \mp u_{2,1}$ and $Tv^*_{1,2} = \mp v_{2,1}$, which implies a spin flip. Concerning the mass sign-inversion one obtains with $\mathcal{M} = M\mathtt{M}$ the result

$$\begin{aligned} \mathcal{M}\psi_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= \psi_{\mathrm{P}1,2}(\mathbf{p}, m; \mathbf{x}, t), \\ \mathcal{M}\psi_{\mathrm{A}1,2}(\mathbf{p}, m; \mathbf{x}, t) &= \psi_{\mathrm{A}1,2}(\mathbf{p}, m; \mathbf{x}, t). \end{aligned} \quad (27)$$

We note that to obtain this result the spinors u and v are understood to depend implicitly also on the mass m, which means that $u(-m) = v(m)$ and vice versa, since they are the eigenvectors of the Dirac equation in the rest frame and correspond to the eigenvalues $E = \pm m$. Here we used further that operation with $M = \gamma_5$ on the basis vectors yields $Mu_{1,2} = -v_{1,2}$, and thus $Mv_{1,2} = -u_{1,2}$. Apparently, the spin is not flipped in this case. As a result, the solutions in Equation (24) simply are the eigenfunctions of the mass sign-inversion operator with the trivial eigenvalue plus one. Conventionally, in the SM the plus sign corresponds to particles with negative frequency waves and the minus sign to antiparticles with positive frequency waves. This remains true also for the extended Dirac equation describing a massive fermion doublet. The fundamental sense of the sign of m is also obvious from the polarization spinors in Equation (24).

Obviously, the effects of the four symmetry transformations yield subtle individual results. However, if we finally combine them in their product $\mathcal{CPTM}$, we obtain a much simpler result for the general Dirac spinor field

$$\mathcal{CPTM}\,\psi(m; \mathbf{x}, t) = \mathtt{CPTM}\,\psi = \psi^*(-m; -\mathbf{x}, -t). \quad (28)$$

In particular for the spinor wave functions in Equation (24) we finally obtain $\mathcal{CPTM}\,\psi_{\mathrm{P,A}}(\mathbf{p}, m; \mathbf{x}, t) = \psi^*_{\mathrm{A,P}}(\mathbf{p}, m; \mathbf{x}, t)$. In conclusion, we suggest to include the mass sign-inversion operator $\mathcal{M}$ as intrinsic symmetry of the Dirac equation in the CPT theorem, which yields the extended CPTM theorem and the rather concise result in Equation (28). The operation of γ_5 has of course no effect on the scalar plane wave and particle momentum as discussed above, but effectively merely acts on the polarization vectors of the Dirac equation. Importantly, it plays a key role in chirality through the projection operator $P_\pm = \frac{1}{2}(1_4 \pm \gamma_5)$, which permits to decompose the spinor field ψ into its left- and right-chiral spinor components.

4. CPTM SYMMETRY IN GENERAL RELATIVITY

It is of interest to study the CPTM transformation in the astronomical or cosmological applications even though the CPTM symmetry is derived for the flat Minkowski space-time. In the gravitational theory (in the frame of general relativity), the charge conjugation C is ineffective, and parity and time reversal act as the PT transformation, $\mathrm{d}x^\mu \to -\mathrm{d}x^\mu$ [16]. The energy-momentum tensor $T^{\mu\nu}$ transforms anti-symmetrically under the CPTM transformation. For a point mass at x_0, for example, the energy-momentum tensor $T^{\mu\nu}$ transforms as

$$\frac{1}{\sqrt{g}} m \int \delta^{(4)}(x - x_0) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \mathrm{d}\tau \\ \to \frac{1}{\sqrt{g}} (-m) \int \delta^{(4)}(x - x_0) \frac{-\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{-\mathrm{d}x^\nu}{\mathrm{d}\tau} \mathrm{d}\tau \tag{29}$$

where $g = |\det(g_{\mu\nu})|$ is the determinant of the metric $g_{\mu\nu}$ in absolute value. The metric tensor $g_{\mu\nu}$, the Ricci curvature tensor $R_{\mu\nu}$, and the scalar curvature R all transform symmetrically under the CPTM transformation in the same way as the CPT transformation (assuming the metric is already given or mass-independent). The Christoffel symbol is anti-symmetric to the CPTM transformation. Using the symmetry properties above, the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{30}$$

is violated in that the l.h.s. (which represents the space-time property) is symmetric to the CPTM transformation while the r.h.s. (which represents the source of gravity) is anti-symmetric. The Einstein equation is CPTM symmetric in vacuum $T_{\mu\nu} = 0$. If, however, the gravitational constant G changes the sign together with the CPTM transformation, the symmetry of the Einstein equation is restored for a non-zero energy-momentum tensor.

The particle equation of motion (represented by the geodesic equation)

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} = -\frac{m_\mathrm{g}}{m_\mathrm{i}} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \Gamma^\lambda_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{31}$$

transforms symmetrically to the CPTM transformation if both the inertial mass m_i and the gravitational mass m_g change the sign under the CPTM transformation. Note that the four-acceleration (l.h.s.) is anti-symmetric. Equation (31) implies that the gravitational interaction between ordinary matter (represented by positive inertial mass) and antimatter (in case it is represented by negative gravitational mass) is repulsive, which according to Villata [16] could be one of the possible mechanisms of large-scale structure formation such as the void regions in galaxy clusters (Mpc or larger).

5. SUMMARY AND CONCLUSION

We have shown that the CPT theorem of the standard Dirac equation also holds for the extended version, which describes a massive fermion doublet obeying the isospin $SU(2)$ symmetry [13]. This result was stated in Equation (16). The combined multiplied matrices of CPT result in Γ_5 which is the key matrix for the definition of chirality in the Dirac equation. Moreover, $\Gamma_5\Phi$ (or $\gamma_5\psi$ for the standard Dirac equation) also provides a solution to it after mass sign reversal (named $\mathtt{M}$), which we suggest to be adopted as another intrinsic symmetry of the Dirac equation. Related to it we therefore propose the symmetry transformation $\mathcal{M} = \Gamma_5\mathtt{M}$. It should be multiplied from the right to CPT, which seems incomplete, but when being upgraded to CPTM it simply is proportional to the unity operator. Therefore, for the standard as well as extended Dirac equation we obtain $\mathcal{CPTM} = 1_4\mathtt{CPTM}$, respectively $\mathcal{CPTM} = 1_8\mathtt{CPTM}$, which yields the fundamental relation Equation (28) for the standard Dirac equation, and similarly

$$\mathcal{CPTM}\,\Phi(m; \mathbf{x}, t) = \mathtt{CPTM}\,\Phi = \Phi^*(-m; -\mathbf{x}, -t) \tag{32}$$

for the extended Dirac equation describing a massive fermion doublet. We recall that the eigenfunctions of the $\mathtt{CPTM}$ operator are just the plane waves [Equation (22)], which represent the wave functions of massive free particles and antiparticles in relativistic quantum theory. With these results we have achieved the main purpose of this short theoretical paper. It emphasizes that by the introduction of the mass inversion operator $\mathcal{M}$ the CPT theorem can be simplified to the above forms for the CPTM theorem, which do not involve Dirac matrices any more but just the related unit matrix, whereas in the usual form of the CPT theorem [4–7] the γ_5 matrix appears explicitly, whereby its physical meaning in this context is according to our knowledge not discussed at all. We speculated [12] about this possible modification of the CPT theorem previously, although we did not evaluate this rigorously as in the present paper.

One natural field of application seems to us the state of unusual non-baryonic matter in the universe. The extension of CPT symmetry to CPTM symmetry may be applied to astrophysical and cosmological contexts and studies, as we already discussed in the introduction. For example, CPTM promotes the notion that negative mass states or particles are one of the likely candidates for dark energy, as was suggested by Petit and d'Agostini [19]. It is pointed out that negative mass particles may exist in the de Sitter space-time like for the inflation phase of the early universe, an issue discussed by Mbarek and Paranjape [20]. Another application would be massive neutrinos. Combination of the Majorana-type mass with the Dirac-type mass in the neutrino field can, after Yanagida [21] and Mohapatra and Senjanović [22], generate neutrino states with negative mass. Other future applications remain to be determined, and the consequences for quantum field theory beyond the SM still need to be investigated.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

EM: concept development, calculations, and manuscript writing. YN: discussion, calculations, literature review, and manuscript writing. Both authors contributed to the article and approved the submitted version.

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APPENDIX: EXTENDED DIRAC MATRICES IN WEYL REPRESENTATION

The standard Dirac matrices can be found and looked up in any textbook [4, 5]. But we first quote them here for completeness in the Weyl representation

$$\gamma^{\mu} = \left(\begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \right), \tag{A1}$$

with the well-known three-vector of the Pauli [23] matrices $\boldsymbol{\sigma} = (\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}})$. According to the new derivations by Marsch and Narita [13], the Dirac matrices for the extended Dirac equation including isospin are based on the generalized 4×4 spin matrices, reading

$$\Sigma^{\pm}_{\mathrm{x}} = \begin{pmatrix} 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$\Sigma^{\pm}_{\mathrm{y}} = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \pm 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},$$

$$\Sigma^{\pm}_{\mathrm{z}} = \begin{pmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix}, \tag{A2}$$

with the commutator $[\boldsymbol{\Sigma}^{\pm}, \boldsymbol{\Sigma}^{\mp}] = 0$. Also, $\boldsymbol{\Sigma}^{\pm} \times \boldsymbol{\Sigma}^{\pm} = 2\mathrm{i}\boldsymbol{\Sigma}^{\pm}$. By complex conjugation of the Sigma matrices in Equation (A2), we can see that they obey $(\boldsymbol{\Sigma}^{\pm})^* = -\boldsymbol{\Sigma}^{\mp}$. The Gamma matrices are defined in the following way

$$\boldsymbol{\Gamma} = \mathrm{i} \begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}^{-} \\ \Delta\boldsymbol{\Sigma}^{+} & 0 \end{pmatrix}. \tag{A3}$$

The involved Sigma matrices and their characteristics were already quoted above. The new Delta matrix corresponds to the metric in Minkowski space-time and is defined as

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A4}$$

which yields $\Delta^2 = 1_4$. Delta has the important property that $\boldsymbol{\Sigma}^{\pm} = \Delta\boldsymbol{\Sigma}^{\mp}\Delta$, which guarantees that $\Gamma_j^2 = -1_8$, and thus $\boldsymbol{\Gamma}^2 = -3\,1_8$, and consequently that the component matrices Γ_j anticommute, and therefore obey the Clifford algebra. The matrices Γ_5 and Γ_0 are given by the expressions

$$\Gamma_5 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix}, \tag{A5}$$

which obeys $(\Gamma_0)^2 = 1_8$ and mutually anticommutes with the other four Gamma matrices by its definition. The spin or spinorial rotation operator is given by

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\Sigma}^{+} & 0 \\ 0 & \boldsymbol{\Sigma}^{-} \end{pmatrix}. \tag{A6}$$

It is interesting to note that charge exchange leads to spin inversion, i.e., flips the spin, namely

$$\mathcal{C}^{-1}\mathbf{S}\mathcal{C} = -\mathbf{S}, \tag{A7}$$

which means these two operators anticommute.