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The Gołab-Schinzel Functional Equation Restricted to Half-Lines

By

Peter Kahlig and **Janusz Matkowski**

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Abstract

Under some regularity conditions, we determine solutions of the functional equation $f(x+yf(x)) = f(x)f(y)$ on domains restricted to half-lines.

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1. Introduction

The Gołab-Schinzel functional equation

$$f(x+yf(x)) = f(x)f(y), \quad x, y \in \mathbb{R},$$

had appeared in connection with determining some subgroups or subsemigroups of the affine group (ST. GOŁAB and A. SCHINZEL [7], cf. also J. ACZÉL and J. DHOMBRES [1], Chapter 19), and was considered by many authors (cf. for instance [2], [4], [5], [8]). It is known that the nontrivial continuous solution $f\colon \mathbb{R} \to \mathbb{R}$ is either of the form

$$f(x) = cx + 1, \quad \text{or} \quad f(x) = \max(cx + 1, 0), \quad x \in \mathbb{R},$$

with a real constant c. Recently, motivated by some applications [9], the solutions of the Gołab-Schinzel equation on restricted domains like $\{(x,y)\colon x \geq 0, y \geq 0\}$ or $\{(x,y)\colon x > 0, y > 0\}$ have been determined ([3], [6], [12], [13], [14]).

The present paper is concerned with a stronger limitation of the variables in the Gołab-Schinzel equation. Restricting this equation to the half-lines

$$\{(x,y)\colon x = 1,\ y = t,\ t > 0\} \quad \text{and} \quad \{(x,y)\colon x = t,\ y = 1,\ t > 0\},$$

and setting

$$b := f(1),$$

we obtain two functional equations in a single variable (of iterative type)

$$f(1 + bt) = bf(t), \tag{1}$$

and

$$f(t + f(t)) = bf(t). \tag{2}$$

Equation (1) is a classical homogeneous linear equation (cf. [10], pp. 58, 106), and equation (2) is a composite equation which for $b = 1$ becomes the well-known Euler equation related to invariant curves (cf. [10], p. 286).

In Section 2 we consider equation (1). Using the general solution, we show that, under some regularity assumptions, the solution of equation (1) is proportional to the basic solution of the original Gołab-Schinzel functional equation.

In Section 3 we deal with equation (2). It turns out that, under some regularity conditions, the solution of equation (2) is the sum of the basic solution of the Gołab-Schinzel equation and a constant.

The most interesting result is given in Section 4. Assume that $b > 1$ and $a > 0$ are fixed. Applying iterative methods, we show that if a function $f\colon (a, \infty) \to (0, \infty)$ is one-to-one in a neighbourhood of infinity and satisfies both equations (1) and (2), then $f(t) = (b-1)t + 1$ for all $t > a$. A generalization of this result is also given.

2. Linear Equation

We begin with the following

Theorem 1. *Let* $b > 0$, $b \neq 1$, *and let*

$$a := \frac{1}{1-b} \quad \textit{if } b < 1,$$

and let $a \geq \frac{1}{1-b}$ *be arbitrarily fixed if* $b > 1$.

A function $f\colon (a,\infty)\to\mathbb{R}$ *satisfies the equation*

$$f(1+bt) = bf(t), \quad t > a, \tag{3}$$

iff there is a log *b-periodic function* $p\colon \mathbb{R}\to\mathbb{R}$ *such that*

$$f(t) = [(b-1)t+1]p\left(\log\left(t-\frac{1}{b-1}\right)\right) \quad \textit{for } t > a;$$

moreover, f *is positive* (*nonnegative*) *iff so is* p.

Proof. Assume first that the function $f\colon (a,\infty)\to\mathbb{R}$ is a solution of equation (3). Since $f_0\colon (a,\infty)\to\mathbb{R}$,

$$f_0(t) := (b-1)t+1$$

satisfies equation (3), the function $g\colon (a,\infty)\to\mathbb{R}$ given by

$$g := \frac{f}{f_0}$$

satisfies the functional equation

$$g(1+bt) = g(t), \quad t > a.$$

(The particular solution f_0 of equation (3) in the case $0 < b < 1$ is negative and in the case $b > 1$ is positive on the interval under consideration, hence $g = f/f_0$ can be formed.) Writing the latter equation in the form

$$g\left(\exp\left(\log\left(t+\frac{1}{b-1}\right)+\log b\right)-\frac{1}{b-1}\right)$$
$$= g\left(\exp\left(\log\left(t+\frac{1}{b-1}\right)\right)-\frac{1}{b-1}\right),$$

we infer that the function p defined by

$$p(u) := g\left(\exp u-\frac{1}{b-1}\right)$$

is log b-periodic, that is,

$$p(u+\log b) = p(u).$$

Since $f(t) = f_0(t)g(t)$, we hence obtain

$$f(t) = [(b-1)t+1]p\left(\log t-\frac{1}{b-1}\right), \quad t > a.$$

It is easy to verify that for an arbitrary log b-periodic function $p\colon \mathbb{R} \to \mathbb{R}$, the function f given by this formula is a solution of equation (3) in (a, ∞). This completes the proof.

Remark 1. *For a fixed $t_0 > \frac{1}{1-b}$, and for every real function f_0 defined on the closed interval with endpoints t_0 and $1 + bt_0$ and such that $f_0(1 + bt_0) = bf_0(t_0)$, there is a unique solution $f\colon (\frac{1}{1-b}, \infty) \to \mathbb{R}$ of equation* (3) *which is an extension of f_0. Moreover, if f_0 is continuous or* (*and*) *monotonic then so is f. Thus the continuous and monotonic solution of equation* (3) *depends on an arbitrary function* (cf. M. Kuczma [10], e.g. sect. III. 4 and sect. V. 3).

Remark 2. *For $b = 1$ equation* (3) *becomes an equation for* 1-*periodic functions.*

As a simple consequence of Theorem 1 we obtain the following

Corollary 1. *Let $b \in (0, 1)$ and $a = \frac{1}{1-b}$ be fixed. If $f\colon (a, \infty) \to [0, \infty)$ is a decreasing solution of equation* (3), *then $f = 0$ in (a, ∞).*

Theorem 2. *Let $b > 1$ and $a \geq \max(0, \frac{1}{1-b})$ be fixed. Suppose that $f\colon (a, \infty) \to \mathbb{R}$ is a solution of equation* (3). *If the function*

$$(a, \infty) \ni t \to \frac{f(t)}{t}$$

is monotonic in (α, ∞) for some $\alpha > a$, or there exists a finite

$$\lim_{t \to \infty} \frac{f(t)}{t},$$

then there is a $c \in \mathbb{R}$, such that

$$f(t) = c[(b-1)t + 1], \quad t > a.$$

Proof. By Theorem 1 there is a log b-periodic function such that

$$\frac{f(t)}{t} = \left((b-1) + \frac{1}{t}\right) p\left(\log t - \frac{1}{b-1}\right), \quad t > a.$$

Since $\lim_{t \to \infty}\left((b-1) + \frac{1}{t}\right) = b - 1$, the assumed monotonicity of this function implies that p must be constant. The remaining part is obvious. This completes the proof.

Applying some well-known properties of convex functions we hence obtain

Corollary 2. *Let $b > 1$ and $a \geq \max(0, \frac{1}{1-b})$ be fixed. If $f\colon (a, \infty) \to \mathbb{R}$ is a solution of equation* (3) *and it is convex or* *concave in an interval* (α, ∞) *for some* $\alpha > a$, *then there is a* $c \in \mathbb{R}$ *such that*

$$f(t) = c[(b-1)t + 1], \quad t > a.$$

Proof. Suppose that f is convex in an interval (α, ∞) for some $\alpha > a$. By the well-known properties of convex functions, the one-sided derivatives f'_- and f'_+ exist in (α, ∞) and are nondecreasing. In view of Theorem 1 we have

$$f(t) = [(b-1)t + 1]p\left(\log t - \frac{1}{b-1}\right), \quad t > a,$$

where p is a log b-periodic function. It follows that p'_- and p'_+ exist too. Denoting by p' one of these one-sided derivatives we infer that

$$f'(t) = (b-1)p\left(\log t - \frac{1}{b-1}\right) + \frac{(b-1)t+1}{t}p'\left(\log t - \frac{1}{b-1}\right)$$

is nondecreasing in (α, ∞). Replacing here t by $e^{t+n\log b}$ and taking into account that p and p' are log b-periodic, we hence obtain that, for every positive integer n, the function

$$f'(e^{t+n\log b}) = (b-1)p\left(t - \frac{1}{b-1}\right) + \frac{(b-1)e^{t+n\log b}+1}{e^{t+n\log b}}p'\left(t - \frac{1}{b-1}\right)$$

is nondecreasing in (α, ∞). Letting $n \to \infty$ we infer that the function $(b-1)(p+p')$ is nondecreasing in (α, ∞). The periodicity of p and p' implies that $p + p'$ is constant. Thus there is a $c \in \mathbb{R}$ such that

$$p + p' = c.$$

Since p' denotes a one-sided derivative, it follows that $p'_- = p'_+$ and, consequently, p is differentiable. Solving the above differential equation for p and taking into account that p is periodic we conclude that p is constant. This completes the proof.

3. Composite Equation

In this section we deal with the composite functional equation (2). We begin with the following

Theorem 3. *Let* $b \in (-1, 1)$, $b \neq 0$, *be fixed. If* $f: \mathbb{R} \to \mathbb{R}$ *is a continuous solution of the functional equation*

$$f(t + f(t)) = bf(t), \quad t \in \mathbb{R},$$

possessing only one zero, then there is a $c \in \mathbb{R}$ *such that*

$$f(t) = (b-1)t + c, \quad t \in \mathbb{R}.$$

Proof. From (2), by induction, we obtain

$$f\left(t + \frac{1-b^n}{1-b} f(t)\right) = b^n f(t), \quad t \in \mathbb{R},$$

for all positive integers n. Letting $n \to \infty$, and taking into account that f is continuous and $|b| < 1$, we get

$$f\left(t + \frac{1}{1-b} f(t)\right) = 0, \quad t \in \mathbb{R}.$$

Denoting by t_0 the only zero of the function f, we have

$$t + \frac{1}{1-b} f(t) = t_0, \quad t \in \mathbb{R}.$$

Setting $c := (1-b)t_0$ we obtain the desired formula.

In the case $b = 1$ equation (2) becomes the well-known Euler functional equation. Recall the following (cf. M. Kuczma [10], p. 286)

Theorem 4 (Kuratowski, Wagner). *The only solutions* $f\colon \mathbb{R} \to \mathbb{R}$ *of the functional equation*

$$f(t + f(t)) = f(t), \quad t \in \mathbb{R},$$

possessing the Darboux property, are constant functions.

In the case $b > 1$ we have the following

Theorem 5. *Let* $b > 1$, *and* $a > 0$ *be fixed. If* $f\colon (a, \infty) \to [0, \infty)$ *is a solution of the functional equation*

$$f(t + f(t)) = bf(t), \quad t > a,$$

and f *is convex or concave in* (α, ∞) *for some* $\alpha > a$, *then there are* $c \in \mathbb{R}$ *and* $\beta \geq \alpha$ *such that*

$$f(t) = (b-1)t + c, \quad t > \beta.$$

If moreover $f\colon (a, \infty) \to (0, \infty)$ *then*

$$f(t) = (b-1)t + c, \quad t > a.$$

Proof. By induction, for all positive integers n, we have

$$f\left(t + \frac{b^n - 1}{b-1} f(t)\right) = b^n f(t), \quad t > a. \tag{4}$$

If $f = 0$ in (a, ∞) then there is nothing to prove. In the opposite case there is a $t_0 > a$ such that $f(t_0) > 0$. Putting

$$t_n := t_0 + \frac{b^n - 1}{b - 1} f(t_0), \quad n \in \mathbb{N},$$

we get

$$f(t_n) = b^n f(t_0), \quad n \in \mathbb{N}.$$

Consequently,

$$\frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n} = \frac{b^{n+1} f(t_0) - b^n f(t_0)}{(t_0 + \frac{b^{n+1}-1}{b-1} f(t_0)) - (t_0 + \frac{b^n-1}{b-1} f(t_0))}, \quad n \in \mathbb{N},$$

which simplifies to the relation

$$\frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n} = b - 1, \quad n \in \mathbb{N}.$$

Since

$$\lim_{n \to \infty} t_n = \infty,$$

the convexity (or concavity) of f in (α, ∞) implies that there is a $\beta \geq \alpha$ such that

$$\frac{f(t) - f(s)}{t - s} = b - 1, \quad s, t \in (\beta, \infty).$$

It follows that there is a $c \in \mathbb{R}$ such that

$$f(t) = (b - 1)t + c, \quad t > \beta. \tag{5}$$

If f is positive then for every $t > a$ there is a positive integer n such that

$$t + \frac{b^n - 1}{b - 1} f(t) > \beta.$$

Now from (4) and (5) we get

$$(b - 1)\left(t + \frac{b^n - 1}{b - 1} f(t)\right) + c = b^n f(t)$$

which simplifies to the desired relation

$$f(t) = (b - 1)t + c.$$

This completes the proof.

Theorem 6. *Let $b > 0$, $b \neq 1$, be fixed. Suppose that either f: $[0, \infty) \to [f(0), \alpha)$ where $\alpha \leq \infty$, or f: $[0, \infty) \to (\alpha, f(0)]$ where $\alpha \geq -\infty$, is a bijective solution of the functional equation*

$$f(t + f(t)) = bf(t), \quad t \geq 0.$$

If f^{-1} is continuous at $f(0)$ then

$$f(t) = (b-1)t + f(0), \quad t \geq 0.$$

Proof. Suppose that f: $[0, \infty) \to [f(0), \alpha)$. The function $g := f^{-1}$ satisfies the functional equation

$$g(bs) = g(s) + s, \quad s \geq f(0),$$

and is right continuous at $f(0)$. Applying Theorem 5.1 in M. KUCZMA [10], p. 106, we obtain

$$g(s) = f(0) + \frac{s}{b-1}, \quad s \geq f(0).$$

Since an argument to show the remaining statement is analogous, this completes the proof.

Similarly, applying Theorem 5.3 in M. KUCZMA [10], p. 108, we can prove

Theorem 7. *Let $b > 0$, $b \neq 1$, be fixed. Suppose that f: $(0, \infty) \to (\alpha, \beta)$ where $-\infty \leq \alpha < \beta \leq \infty$. If f is a homeomorphic solution of the functional equation*

$$f(t + f(t)) = bf(t), \quad t > 0,$$

then

$$f(t) = (b-1)t + f(0), \quad t > 0.$$

4. A System of Functional Equations

From the point of view of the theory of the Gołab-Schinzel functional equation the most interesting result of this paper reads as follows.

Theorem 8. *Let $b>1$, and $a>0$ be fixed. Suppose that f: $(a, \infty) \to (0, \infty)$ satisfies the pair of functional equations*

$$f(1 + bt) = bf(t), \qquad f(t + f(t)) = bf(t), \quad t > a. \tag{6}$$

If f is one-to-one in a neighbourhood of ∞, then

$$f(t) = (b-1)t + 1, \quad t > a.$$

Proof. Suppose that $f\colon (a,\infty)\to(0,\infty)$ satisfying system (6) is one-to-one in an interval (β,∞) for some $\beta\geq a$. Iterating the first equation we obtain

$$f\left(\sum_{k=0}^{n} b^k + b^{n+1}t\right) = b^{n+1}f(t), \quad t>a,\ n\in\mathbb{N}.$$

Similarly, iterating the second equation, we get

$$f\left(t+\left(\sum_{k=0}^{n} b^k\right)f(t)\right) = b^{n+1}f(t), \quad t>a,\ n\in\mathbb{N}.$$

Both these equations imply that

$$f\left(\sum_{k=0}^{n} b^k + b^{n+1}t\right) = f\left(t+\left(\sum_{k=0}^{n} b^k\right)f(t)\right), \quad t>a,\ n\in\mathbb{N}. \tag{7}$$

Take an arbitrary $t>a$. Since $f(t)>0$ and $b>1$ there is a positive integer n such that

$$\sum_{k=0}^{n} b^k + b^{n+1}t > \beta, \qquad t+\left(\sum_{k=0}^{n} b^k\right)f(t) > \beta.$$

Now the injectivity of f in (β,∞) and relation (7) imply that

$$\sum_{k=0}^{n} b^k + b^{n+1}t = t+\left(\sum_{k=0}^{n} b^k\right)f(t),$$

which simplifies to the relation

$$f(t) = (b-1)t+1,$$

as desired. The proof is completed.

Now, replacing the injectivity condition by a much weaker one, we prove a generalization of the previous result.

Theorem 9. *Let* $b>1$, *and* $a>0$ *be fixed. Suppose that* $f\colon (a,\infty)\to(0,\infty)$ *satisfies the pair of functional equations* (6). *If there exist* $\alpha\geq a$ *and* $M>0$ *such that, for all* $t_1,t_2>\alpha$,

$$f(t_1)=f(t_2) \Rightarrow |t_1-t_2|\leq M,$$

then

$$f(t) = (b-1)t+1, \quad t>a.$$

Proof. Let us fix arbitrarily $t > a$ and put

$$t_{1,n} := \sum_{k=0}^{n} b^k + b^{n+1} t, \qquad t_{2,n} := t + \left(\sum_{k=0}^{n} b^k \right) f(t).$$

According to (7) we have

$$f(t_{1,n}) = f(t_{2,n}), \quad n \in \mathbb{N}.$$

As

$$\lim_{n\to\infty} t_{1,n} = \infty = \lim_{n\to\infty} t_{2,n},$$

for sufficiently large n, we have $t_{1,n}$, $t_{2,n} > \alpha$. From the assumed implication we obtain that, for all sufficiently large $n \in \mathbb{N}$,

$$|t_{2,n} - t_{1,n}| = \frac{b^{n+1} - 1}{b - 1} |f(t) - 1 - (b-1)t| \leq M.$$

Since $\lim_{n\to\infty} b^{n+1} = \infty$, we infer that

$$f(t) - 1 - (b-1)t = 0,$$

which completes the proof.

5. Final Remarks

Restricting the Gołab-Schinzel functional equation to the pair of half-lines $\{(x,y)\colon x = a,\ y > 0\}$ and $\{(x,y)\colon x > 0,\ y = a\}$ where $a > 0$ is fixed, and assuming $b := f(a)$, one gets the pair of iterative functional equations

$$f(a + bt) = bf(t), \qquad f(t + af(t)) = bf(t), \quad t > 0.$$

The iteration procedure applied to each of these equations leads to two infinite systems of equations,

$$f\left(\sum_{k=0}^{n} b^k + b^{n+1} t \right) = b^{n+1} f(t), \quad t > a,\ n \in \mathbb{N},$$

and

$$f\left(t + \left(\sum_{k=0}^{n} b^k \right) f(t) \right) = b^{n+1} f(t), \quad t > a,\ n \in \mathbb{N}.$$

Using these systems one can prove analogous results as above.

Let us mention that the system of iterative functional equations which appears when the Gołab-Schinzel functional equation is restricted to the two parallel lines $x = a$ and $x = b$ was considered in [11].

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Authors' addresses: Peter Kahlig, c/o Institute of Meteorology and Geophysics, University of Vienna, UZA2, 1090 Vienna, Austria, *E-Mail address*: Peter.Kahlig@univie.ac.at; Janusz Matkowski, Institute of Mathematics, University of Zielona Góra, Podgórna 50, 65246 Zielona Góra, Poland, *E-Mail address*: J.Matkowski@im.uz.zgora.pl