

Stability of the Homogeneity and Completeness

By

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Abstract

It is known that the classical homogeneity equation $f(\alpha x) = \alpha f(x)$ under rather weak conditions is stable (cf. [4, 8]). Following J. SCHWAIGER [14] we will consider here the stability of the ϕ -homogeneity equation $f(\alpha x) = \phi(\alpha)f(x)$. We will prove, without assumption that ϕ is a homomorphism, the stability of the ϕ -homogeneity equation under some conditions.

One of the necessary conditions is that the target space of the function f be sequentially complete. We will also prove that the converse is true, that is we will show that stability of the ϕ -homogeneity implies the fact that if the target space is a normed space then it has to be a Banach space.

1. Introduction

Since the time when ULAM ([17]) had posed his famous problem many authors have considered stability of different functional equations (cf. [4, 8]). Among others, in 1992 CZERWIK ([1]) examined the stability of the v -homogeneity equation

$$F(\alpha x) = \alpha^v F(x) \quad \text{for} \quad \alpha \in U_v, \quad x \in X,$$

where a function F maps a real linear space X into a real Banach space Y , $v \in \mathbb{R} \setminus \{0\}$ is fixed and $U_v := \{\alpha \in \mathbb{R}: \alpha^v \text{ is well defined}\}$. It has been proved there that if a function $f: X \rightarrow Y$ satisfies the inequality

$$\|f(\alpha x) - \alpha^v f(x)\| \leq g(\alpha, x) \quad \text{for} \quad \alpha \in U_v, \quad x \in X,$$

with a function $g: U_v \times X \rightarrow [0, \infty)$ then under some assumptions on g there exists a v -homogeneous function $F: X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq h(x) \quad \text{for } x \in X,$$

with some function $h: X \rightarrow [0, \infty)$ depending on g .

Independently, JÓZEF TABOR proved in [16] that every mapping f from a real linear space X into a normed space Y satisfying

$$\|\alpha^{-1}f(\alpha x) - f(x)\| \leq \varepsilon \quad \text{for } \alpha \in \mathbb{R} \setminus \{0\}, \quad x \in X,$$

where $\varepsilon \geq 0$ is given, is homogeneous. These results then were generalized successively in different directions.

In [15] the inequality

$$\|f(\alpha x) - \alpha^v f(x)\| \leq g(\alpha, x), \quad \alpha \in \mathbb{R} \setminus \{0\}, \quad x \in X,$$

with a constant $v \in \mathbb{R} \setminus \{0\}$ and a function g mapping $\mathbb{R} \times X$ into \mathbb{R} has been investigated.

KOMINEK and MATKOWSKI began to investigate in [11] the stability of the homogeneity on a restricted domain. They have considered the condition

$$\alpha^{-1}f(\alpha x) - f(x) \in V, \quad \alpha \in A, \quad x \in S,$$

for the mapping f from a cone $S \subset X$ into a sequentially complete locally convex linear topological Hausdorff space Y over \mathbb{R} and a subset $A \subset (1, \infty)$. This result has been generalized in [9] and [14]. SCHWAIGER [14] has examined the condition

$$f(\alpha x) - \phi(\alpha)f(x) \in V(\alpha), \quad \text{for } \alpha \in A, \quad x \in X,$$

where

- G is a semigroup with unit acting on the non-empty set X ;
- Y is a sequentially complete locally convex linear topological Hausdorff space Y over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$;
- $A \subset G$ generates G as a semigroup;
- $\phi: G \rightarrow \mathbb{K}$ is a function;
- $V: G \rightarrow \mathcal{B}(Y)$ is a mapping from G into the family $\mathcal{B}(Y)$ of all bounded subsets of Y .

It is proved there that if $f(X)$ is unbounded, then ϕ is a multiplicative function, i.e. ϕ satisfies the equation

$$\phi(\alpha\beta) = \phi(\alpha)\phi(\beta) \quad \text{for } \alpha, \beta \in G,$$

and there is a function $F: X \rightarrow Y$ satisfying

$$F(\alpha x) = \phi(\alpha)F(x) \quad \text{for} \quad \alpha \in G, \quad x \in X,$$

(we say then that F is ϕ -homogeneous) and such that the difference $F - f$ is suitably bounded on X .

2. Notations

We begin here with notations we will need in the following. A semigroup (G, \cdot) we will call a *semigroup with zero* if there exists an element $0 \in G$ such that $0 \cdot \alpha = \alpha \cdot 0 = 0$ for every $\alpha \in G$. A semigroup (G, \cdot) we will call a *monoid* (cf. [6]) if there exists $1 \in G$ such that $1 \cdot \alpha = \alpha \cdot 1 = \alpha$ for $\alpha \in G$. Furthermore, a triple $(G, \cdot, 0)$ we will call a *group with zero* if (G, \cdot) is a monoid with zero and every element of the set $G^* := G \setminus \{0\}$ is invertible, that is for each $g \in G^*$ there is $g^{-1} \in G^*$ with $g \cdot g^{-1} = g^{-1} \cdot g = 1$. A subset $A \subset G$ of a group with zero $(G, \cdot, 0)$ we will call a *subgroup of the group G* if $A^* \neq \emptyset$ and A^* is a subgroup of the group G^* . As it is easy to see a set A is a subgroup of the group with zero G if and only if $A^* \neq \emptyset$ and $\alpha\beta^{-1} \in A$ for $\alpha \in A$ and $\beta \in A^*$.

In the following lemma we list some properties of homomorphisms of groups with zeros. Since they are analogous to similar properties of multiplicative functions on the real line, we omit the proof of them.

Lemma 1. *Let $(G, \cdot, 0)$ and $(H, \cdot, 0)$ be groups with zeros and let $\phi: G \rightarrow H$ be a homomorphism, i.e. ϕ satisfies the equation*

$$\phi(\alpha\beta) = \phi(\alpha)\phi(\beta) \quad \text{for} \quad \alpha, \beta \in G.$$

Then

- (i) $\phi(0) \in \{0, 1\}$;
- (ii) if $\phi(0) = 1$ then $\phi = 1$;
- (iii) if $\phi \neq 1$ then $\phi(0) = 0$;
- (iv) if $\phi(\alpha_0) = 0$ for some $\alpha_0 \in G^*$ then $\phi = 0$;
- (v) if $\phi \neq 0$ then $\phi(1) = 1$.

Let (G, \cdot) be a semigroup and let $\emptyset \neq A \subset G$. By $\langle A \rangle_s$ we denote the *subsemigroup of G generated by A* , whereas if $(G, \cdot, 0)$ is a group with zero then by $\langle A \rangle$ we denote the *subgroup of G generated by A* . As one can see we have

$$\langle A \rangle_s := \left\{ \prod_{i=1}^n \alpha_i : \alpha_i \in A, i \in \{1, \dots, n\}, n \in \mathbb{N} \right\},$$

$$\langle A \rangle := \left\{ \prod_{i=1}^n \alpha_i^{\varepsilon_i} : \varepsilon_i \in \{-1, 1\}, \alpha_i \in A \text{ if } \varepsilon_i = 1, \alpha_i \in A^* \text{ if } \varepsilon_i = -1, n \in \mathbb{N} \right\}.$$

In the case where $(G, \cdot, 0)$ is Abelian, for a set $A \subset G, A^* \neq \emptyset$, we have (see [12], Theorem 1, p. 89) $\langle A \rangle = \langle A \rangle_s \langle A^* \rangle_s^{-1}$. Then in virtue of Theorem 1 ([12], p. 471), we get

Lemma 2. *Let $(G, \cdot, 0)$ and $(H, \cdot, 0)$ be Abelian groups with zeros and let $A \subset G$ be a subsemigroup such that $A^* \neq \emptyset$. Assume that $\phi: A \rightarrow H$ is a homomorphism such that $\phi(A^*) \subset H^*$. Then there exists exactly one homomorphism $\tilde{\phi}: \langle A \rangle \rightarrow H$ such that $\tilde{\phi}|_A = \phi$.*

Finally, we will need the notion of a G -space. We will use here a generalization of the standard notion of a G -space to the semigroup case (cf. [13, I, § 5, pp. 25–33]).

Definition 1. Let (G, \cdot) be a semigroup and let X be a nonempty set with a fixed element θ which we will call zero. Assume that we are given a semigroup action on X , that is we have a function $\cdot: G \times X \rightarrow X$ which satisfies the following conditions:

$$\begin{aligned} (g_1 g_2)x &= g_1(g_2x) & \text{for } g_1, g_2 \in G, \quad x \in X, \\ 1x &= x & \text{for } x \in X, \quad \text{if } 1 \in G. \end{aligned}$$

Let moreover

$$\begin{aligned} g\theta &= \theta & \text{for } g \in G, \\ 0x &= \theta & \text{for } x \in X, \quad \text{if } 0 \in G. \end{aligned}$$

Then the pair (X, G) satisfying these conditions we will call a G -space.

3. The Homogeneity Equation

In the following we assume that (G, \cdot) and (H, \cdot) are semigroups and that (X, G) and (Y, H) are G - and H -spaces, respectively. Let $\emptyset \neq A \subset G$ and assume that $\emptyset \neq U \subset X$ is a set such that $AU \subset U$ (we have then $\langle A \rangle_s U \subset U$). We will consider here solutions $\phi: A \rightarrow H$ and $F: U \rightarrow Y$ of the ϕ -homogeneity equation

$$F(\alpha x) = \phi(\alpha)F(x) \quad \text{for } \alpha \in A, \quad x \in U. \tag{1}$$

We will assume additionally that

the mapping $H \ni \alpha \mapsto \alpha x \in Y$ is injective for every $x \in Y^* := Y \setminus \{\theta\}$.

As one can easily see later, this assumption is crucial only in the case when we will not assume that ϕ is a homomorphism. On the other hand, we will use the results of this section in the case when the target

space for the function F is a linear space and then this condition is trivially fulfilled.

At first we consider the case when there exists a homomorphism $\tilde{\phi}: \langle A \rangle_s \rightarrow H$ such that $\tilde{\phi}|_A = \phi$. We have then

Lemma 3. *Assume that a homomorphism $\tilde{\phi}: \langle A \rangle_s \rightarrow H$ and a function $F: U \rightarrow Y$ satisfy the equation*

$$F(\alpha x) = \tilde{\phi}(\alpha)F(x) \quad \text{for} \quad \alpha \in A, \quad x \in U. \quad (2)$$

Then

$$F(\alpha x) = \tilde{\phi}(\alpha)F(x) \quad \text{for} \quad \alpha \in \langle A \rangle_s, \quad x \in U. \quad (3)$$

Proof. Let $\alpha = \prod_{i=1}^n \alpha_i \in \langle A \rangle_s$, $\alpha_i \in A$ for $i = 1, \dots, n$ with some $n \in \mathbb{N}$. Then for arbitrary $x \in U$, from (2) we get

$$\begin{aligned} F(\alpha x) &= F\left(\left(\prod_{i=1}^n \alpha_i\right)x\right) = \left(\prod_{i=1}^n \tilde{\phi}(\alpha_i)\right)F(x) \\ &= \tilde{\phi}\left(\prod_{i=1}^n \alpha_i\right)F(x) = \tilde{\phi}(\alpha)F(x). \quad \square \end{aligned}$$

Now we will show that in some cases a function ϕ satisfying jointly with F Eq. (1) can be extended to a homomorphism $\tilde{\phi}$ such that the functions $\tilde{\phi}$ and F will satisfy the $\tilde{\phi}$ -homogeneity equation (3).

Theorem 1. *Assume that the functions $\phi: A \rightarrow H$ and $F: U \rightarrow Y$ satisfy Eq. (1). Then either $F = \theta$ or there exists exactly one homomorphism $\tilde{\phi}: \langle A \rangle_s \rightarrow H$ such that $\tilde{\phi}|_A = \phi$ and the function F satisfies then the $\tilde{\phi}$ -homogeneity equation (3).*

Proof. Assume that $F(x_0) \neq \theta$ for some $x_0 \in U$. Fix $\alpha \in \langle A \rangle_s$ arbitrarily. Then there are $n \in \mathbb{N}$ and $\lambda_i \in A$ for $i = 1, \dots, n$ such that $\alpha = \prod_{i=1}^n \lambda_i$. Put

$$\tilde{\phi}(\alpha) := \prod_{i=1}^n \phi(\lambda_i).$$

We will show that $\tilde{\phi}$ is well defined. Let $\alpha = \prod_{i=1}^n \lambda_i = \prod_{j=1}^m \nu_j$ with some $\lambda_i, \nu_j \in A$ for $i = 1, \dots, n, j = 1, \dots, m$, where $n, m \in \mathbb{N}$. From (1) we get then

$$\begin{aligned} \phi(\lambda_1) \cdots \phi(\lambda_n)F(x_0) &= F(\lambda_1 \cdots \lambda_n x_0) \\ &= F(\nu_1 \cdots \nu_m x_0) = \phi(\nu_1) \cdots \phi(\nu_m)F(x_0). \end{aligned}$$

The mapping $H \ni \alpha \mapsto \alpha F(x_0) \in Y$ is injective, so

$$\prod_{i=1}^n \phi(\lambda_i) = \prod_{j=1}^m \phi(\nu_j).$$

From the definition of $\tilde{\phi}$ we have $\tilde{\phi}|_A = \phi$. Moreover $\tilde{\phi}$ is a homomorphism. Indeed, for $\alpha = \prod_{i=1}^n \lambda_i, \beta = \prod_{j=1}^m \nu_j \in \langle A \rangle_s$ with some $\lambda_i, \nu_j \in A$ for $i = 1, \dots, n, j = 1, \dots, m$, where $n, m \in \mathbb{N}$, we have

$$\tilde{\phi}(\alpha\beta) = \tilde{\phi}\left(\prod_{i=1}^n \lambda_i \prod_{j=1}^m \nu_j\right) = \prod_{i=1}^n \phi(\lambda_i) \prod_{j=1}^m \phi(\nu_j) = \tilde{\phi}(\alpha)\tilde{\phi}(\beta).$$

Finally we show that $\tilde{\phi}$ is unique. Suppose that $\tilde{\phi}_1, \tilde{\phi}_2: \langle A \rangle_s \rightarrow H$ are homomorphisms such that $\tilde{\phi}_1|_A = \tilde{\phi}_2|_A = \phi$. Fix $\alpha \in \langle A \rangle_s$. Then there exist $\lambda_i \in A$ for $i = 1, \dots, n, n \in \mathbb{N}$ such that $\alpha = \prod_{i=1}^n \lambda_i$. Thus

$$\begin{aligned} \tilde{\phi}_1(\alpha) &= \tilde{\phi}_1\left(\prod_{i=1}^n \lambda_i\right) = \prod_{i=1}^n \tilde{\phi}_1(\lambda_i) \\ &= \prod_{i=1}^n \tilde{\phi}(\lambda_i) = \prod_{i=1}^n \tilde{\phi}_2(\lambda_i) = \tilde{\phi}_2\left(\prod_{i=1}^n \lambda_i\right) = \tilde{\phi}_2(\alpha). \end{aligned}$$

Hence from (1) we obtain

$$F(\alpha x) = \tilde{\phi}(\alpha)F(x) \quad \text{for} \quad \alpha \in A, \quad x \in U.$$

Then Lemma 3 finishes the proof. \square

In the following we will assume that $(G, \cdot, 0)$ and $(H, \cdot, 0)$ are Abelian groups with zeros. Moreover let (X, G) and (Y, H) be G - and H -spaces, respectively. Let $A \subset G$ be such that $A^* \neq \emptyset$ and let $U \subset X, AU \subset U$. As in the semigroup case we begin with the case when there exists a homomorphism $\phi: \langle A \rangle \rightarrow H$ such that $\phi|_A = \phi$. Then in virtue of Lemma 3, from (1) we get

$$F(\alpha x) = \tilde{\phi}(\alpha)F(x) \quad \text{for} \quad \alpha \in \langle A \rangle_s, \quad x \in U. \quad (4)$$

Assume that $\tilde{\phi} = 0$. Then from (4) we obtain

$$F(\alpha x) = 0 \quad \text{for} \quad \alpha \in \langle A \rangle_s, \quad x \in U, \quad (5)$$

and then one can easily get $F|_{AU} = 0$.¹

Now let $\tilde{\phi} \neq 0$. Since $\tilde{\phi}$ is a homomorphism so (cf. Lemma 1 (iv)) $\tilde{\phi}(\langle A^* \rangle) \subset H^*$.

¹ In this case we have $AU = \langle A \rangle_s U$.

Theorem 2. *Assume that a nonzero homomorphism $\tilde{\phi}: \langle A \rangle \rightarrow H$ and a function $F: U \rightarrow Y$ satisfy Eq. (4). Then there exists a unique function $\bar{F}: \langle A \rangle U \rightarrow Y$ such that $\bar{F}|_U = F$ and*

$$\bar{F}(\alpha x) = \tilde{\phi}(\alpha)\bar{F}(x) \quad \text{for} \quad \alpha \in \langle A \rangle, \quad x \in \langle A \rangle U. \quad (6)$$

Proof. Fix $x \in \langle A \rangle U$. Then $x = \gamma u \in \langle A \rangle U$ with some $\gamma \in \langle A \rangle$ and $u \in U$. Put

$$\bar{F}(x) := \tilde{\phi}(\gamma)F(u).$$

We will show that \bar{F} is well defined. Let $\gamma_1 u_1 = \gamma_2 u_2 \in \langle A \rangle U$. Since $\langle A \rangle = \langle A \rangle_s \cdot \langle A^* \rangle_s^{-1}$ so there are elements $\lambda_1, \lambda_2 \in \langle A \rangle_s$ and $\eta_1, \eta_2 \in \langle A^* \rangle_s$ such that $\gamma_i = \lambda_i \eta_i^{-1}$ for $i = 1, 2$. Since $\lambda_1 \eta_1^{-1} u_1 = \lambda_2 \eta_2^{-1} u_2$ so (G is an Abelian group) $\lambda_1 \eta_2 u_1 = \eta_1 \lambda_2 u_2$. From (4) we get then

$$\tilde{\phi}(\lambda_1)\tilde{\phi}(\eta_2)F(u_1) = F(\lambda_1 \eta_2 u_1) = F(\eta_1 \lambda_2 u_2) = \tilde{\phi}(\eta_1)\tilde{\phi}(\lambda_2)F(u_2).$$

Hence we obtain

$$\tilde{\phi}(\lambda_1)\tilde{\phi}(\eta_1)^{-1}F(u_1) = \tilde{\phi}(\lambda_2)\tilde{\phi}(\eta_2)^{-1}F(u_2).$$

Then using the fact that $\tilde{\phi}$ is a homomorphism we get

$$\begin{aligned} \tilde{\phi}(\gamma_1)F(u_1) &= \tilde{\phi}(\lambda_1)\tilde{\phi}(\eta_1)^{-1}F(u_1) = \tilde{\phi}(\lambda_2)\tilde{\phi}(\eta_2)^{-1}F(u_2) \\ &= \tilde{\phi}(\gamma_2)F(u_2), \end{aligned}$$

and therefore the function \bar{F} is well defined. From the definition of \bar{F} we obtain that $\bar{F}|_U = F$ (cf. Lemma 1(v)). We will show now that then Eq. (6) is fulfilled. Fix $\alpha \in \langle A \rangle$ and $x \in \langle A \rangle U$. Then $x = \gamma u$ with $\gamma \in \langle A \rangle$, $u \in U$ and $\alpha \gamma \in \langle A \rangle$. Next $\tilde{\phi}$ is a homomorphism, so

$$\bar{F}(\alpha x) = \bar{F}(\alpha \gamma u) = \tilde{\phi}(\alpha \gamma)F(u) = \tilde{\phi}(\alpha)\tilde{\phi}(\gamma)F(u) = \tilde{\phi}(\alpha)\bar{F}(x).$$

Finally we will prove that \bar{F} is unique. Suppose that $F_1, F_2: \langle A \rangle U \rightarrow Y$ are such that $F_1|_U = F_2|_U = F$ and

$$F_i(\alpha x) = \tilde{\phi}(\alpha)F_i(x) \quad \text{for} \quad \alpha \in \langle A \rangle, \quad x \in \langle A \rangle U, \quad i = 1, 2.$$

Fix $x \in \langle A \rangle U$ arbitrarily. Then one can find elements $\gamma \in \langle A \rangle$ and $u \in U$ such that $x = \gamma u$. Thus

$$\begin{aligned} F_1(x) &= F_1(\gamma u) = \tilde{\phi}(\gamma)F_1(u) = \tilde{\phi}(\gamma)F(u) \\ &= \tilde{\phi}(\gamma)F_2(u) = F_2(\gamma u) = F_2(x), \end{aligned}$$

which finishes the proof. □

Finally we obtain

Theorem 3. *Let $\phi: A \rightarrow H$ and $F: U \rightarrow Y$ satisfy (1). Assume additionally that $\phi(\langle A^* \rangle) \subset H^*$. Then either $F = \theta$ or there is a unique homomorphism $\tilde{\phi}: \langle A \rangle \rightarrow H$ and a unique function $\bar{F}: \langle A \rangle U \rightarrow Y$ such that $\tilde{\phi}_A = \phi$, $\bar{F}|_U = F$ and*

$$\bar{F}(\alpha x) = \tilde{\phi}(\alpha)\bar{F}(x) \quad \text{for} \quad \alpha \in \langle A \rangle, \quad x \in \langle A \rangle U. \quad (7)$$

Proof. Assume that ϕ and F satisfy (1). Let $x_0 \in U$ be such that $F(x_0) \neq \theta$. By Theorem 1 there exists a unique homomorphism $\phi_1: \langle A \rangle_s \rightarrow H$ such that $\phi_1|_A = \phi$ and

$$F(\alpha x) = \phi_1(\alpha)F(x) \quad \text{for} \quad \alpha \in \langle A \rangle_s, \quad x \in U. \quad (8)$$

It follows from the definition of the homomorphism ϕ_1 that $\phi_1(\langle A^* \rangle_s) \subset H^*$. Then from Lemma 2 we get that there exists a unique homomorphism $\tilde{\phi}: \langle A \rangle \rightarrow H$ such that $\tilde{\phi}|_{\langle A \rangle_s} = \phi_1$. Theorem 2 then finishes the proof. \square

4. Stability of Homogeneity

From now on let Y stand for a locally convex and sequentially complete linear topological Hausdorff space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Let $V \subset Y$ be a nonempty set. By $\text{aconv } V$ we denote the absolutely convex hull of the set V , by V^β we denote the smallest balanced superset of V , whereas $\text{seq cl } V$ will denote the sequential closure of V . By $\mathcal{B}(Y)$ we denote the family of all bounded subsets of Y .

We will need several properties of convex sets and bounded sets, namely

Lemma 4. (i) *Let $\alpha \in \mathbb{K}$ and $V, V_1, V_2 \in \mathcal{B}(Y)$. Then*

$$\alpha V, \text{cl } V, \text{conv } V, V^\beta, V_1 + V_2 \in \mathcal{B}(Y).$$

In particular, $\text{seq cl } V \in \mathcal{B}(Y)$.

(ii) *The set $\text{conv}(V^\beta)$ is the smallest absolutely convex set containing the set V , i.e. $\text{aconv } V = \text{conv}(V^\beta)$. Thus if $V \in \mathcal{B}(Y)$, then also $\text{aconv } V \in \mathcal{B}(Y)$.*

(iii) *Let $V \subset Y$ be convex (absolutely convex). Then the set $\text{seq cl } V$ is convex (absolutely convex).*

(iv) *If $V \subset Y$ is absolutely convex then for every $\alpha \in \mathbb{K}$ we have $\alpha V = |\alpha|V$.*

(v) *Let $V \subset Y$ and assume that $\alpha, \beta \in \mathbb{K}$ are such that $|\alpha| \leq |\beta|$. Then $\alpha V \subset |\beta| \text{aconv } V$.*

(vi) *Assume that $V \in \mathcal{B}(Y)$, $\alpha_n \geq 0$, $\alpha_n \rightarrow 0$ and $x_n \in \alpha_n V$. Then $x_n \rightarrow 0$.*

(For facts about locally convex spaces consult for example [10].)

4.1. The Semigroup Case

Let (G, \cdot) be a semigroup. By $Z(G)$ we denote the center of the semigroup G . Let (X, G) be a G -space. Assume that $A \subset G$ and $U \subset X$ are nonempty sets such that $AU \subset U$. Let $V: A \rightarrow \mathcal{B}(Y)$, and let $\psi: \langle A \rangle_s \rightarrow [0, \infty)$ be a homomorphism into the multiplicative semigroup $[0, \infty)$ and assume that a function $K: U \rightarrow \mathbb{K}$ satisfies the inequality

$$|K(\alpha x)| \leq \psi(\alpha)|K(x)| \quad \text{for all } \alpha \in \langle A \rangle_s, \quad x \in U. \quad (9)$$

Definition 2. A function $f: U \rightarrow Y$ we will call K -bounded if there exists a bounded set $W \in \mathcal{B}(Y)$ such that $f(x) \in K(x)W$ for every $x \in U$. If not, then f will be called K -unbounded.

We have the following theorem.

Theorem 4. Assume that the functions $\phi: A \rightarrow \mathbb{K}$ and $f: U \rightarrow Y$ satisfy the condition

$$f(\alpha x) - \phi(\alpha)f(x) \in K(x)V(\alpha) \quad \text{for } \alpha \in A, \quad x \in U, \quad (10)$$

where K satisfies (9). If

$$A_1 := \{\alpha \in Z(G) \cap A: |\phi(\alpha)| > \psi(\alpha)\} \neq \emptyset,$$

then there exists a unique function $F: U \rightarrow Y$ such that

$$F(\alpha x) = \phi(\alpha)F(x) \quad \text{for } \alpha \in A, \quad x \in U,$$

and

$$F(x) - f(x) \in K(x)V_0, \quad x \in U,$$

where

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y).$$

Proof. Fix $\alpha_0 \in A_1$ (then $\psi(\alpha_0)/|\phi(\alpha_0)| < 1$). Thus for $m, n \in \mathbb{N}_0$ and $x \in U$ we have

$$\begin{aligned} & \phi(\alpha_0)^{-(m+n)}f(\alpha_0^{m+n}x) - \phi(\alpha_0)^{-m}f(\alpha_0^m x) \\ &= \sum_{k=1}^n \phi(\alpha_0)^{-(m+k)} [f(\alpha_0 \alpha_0^{m+k-1}x) - \phi(\alpha_0)f(\alpha_0^{m+k-1}x)] \\ & \in \sum_{k=1}^n \phi(\alpha_0)^{-(m+k)} K(\alpha_0^{m+k-1}x)V(\alpha_0). \end{aligned}$$

Hence from (9), in virtue of Lemma 4 we get

$$\begin{aligned}
& \sum_{k=1}^n \phi(\alpha_0)^{-(m+k)} K(\alpha_0^{m+k-1}x) V(\alpha_0) \\
& \subset \sum_{k=1}^n |\phi(\alpha_0)|^{-(m+k)} |K(\alpha_0^{m+k-1}x)| \text{aconv } V(\alpha_0) \\
& \subset \sum_{k=1}^n |\phi(\alpha_0)|^{-(m+k)} \psi(\alpha_0^{m+k-1}) |K(x)| \text{aconv } V(\alpha_0) \\
& = \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^m \frac{1}{|\phi(\alpha_0)| - \psi(\alpha_0)} K(x) \left(1 - \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^n \right) \\
& \quad \cdot \text{aconv } V(\alpha_0) \\
& \subset \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^m \frac{1}{|\phi(\alpha_0)| - \psi(\alpha_0)} K(x) \text{aconv } V(\alpha_0).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \phi(\alpha_0)^{-(m+n)} f(\alpha_0^{m+n}x) - \phi(\alpha_0)^{-m} f(\alpha_0^m x) \\
& \in \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^m \frac{1}{|\phi(\alpha_0)| - \psi(\alpha_0)} K(x) \text{aconv } V(\alpha_0). \quad (11)
\end{aligned}$$

Hence, because the set $\text{aconv } V(\alpha_0)$ is bounded we have that for every $x \in U$, $(\phi(\alpha_0)^{-n} f(\alpha_0^n x) : n \in \mathbb{N})$ is a Cauchy sequence. Then the function $F_{\alpha_0} : U \rightarrow Y$,

$$F_{\alpha_0}(x) := \lim_{n \rightarrow \infty} \phi(\alpha_0)^{-n} f(\alpha_0^n x),$$

is well defined. We will show that F_{α_0} satisfies the equation

$$F_{\alpha_0}(\alpha x) = \phi(\alpha) F_{\alpha_0}(x) \quad \text{for } \alpha \in A, \quad x \in U. \quad (12)$$

Put in (10) $\alpha_0^n x$ in the place of x . Then we get

$$f(\alpha \alpha_0^n x) - \phi(\alpha) f(\alpha_0^n x) \in K(\alpha_0^n x) V(\alpha) \subset \psi(\alpha_0)^n K(x) \text{aconv } V(\alpha).$$

Since $\phi(\alpha_0) \neq 0$ and $\alpha_0 \in Z(G)$ (then also $\alpha_0^n \in Z(G)$), so we obtain

$$\phi(\alpha_0)^{-n} f(\alpha_0^n \alpha x) - \phi(\alpha) \phi(\alpha_0)^{-n} f(\alpha_0^n x) \in \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^n K(x) \text{aconv } V(\alpha),$$

and when n tends to infinity and using the fact that the set $\text{aconv } V(\alpha)$ is bounded, from Lemma 4(v) we obtain (12).

From (11), for $m = 0$ we get

$$\phi(\alpha_0)^{-n}f(\alpha_0^n x) - f(x) \in \frac{1}{|\phi(\alpha_0)| - \psi(\alpha_0)} K(x) \text{aconv } V(\alpha_0),$$

and hence

$$F_{\alpha_0}(x) - f(x) \in \frac{1}{|\phi(\alpha_0)| - \psi(\alpha_0)} K(x) \text{seqclaconv } V(\alpha_0) \quad \text{for } x \in U. \quad (13)$$

Put $F := F_{\alpha_0}$. We will show that F is unique. Indeed, suppose that $F_1, F_2: U \rightarrow Y$ satisfy

$$F_i(\alpha x) = \phi(\alpha)F_i(x) \quad \text{for } \alpha \in A, \quad x \in U, \quad i = 1, 2,$$

and

$$F_i(x) - f(x) \in K(x)V_i, \quad \text{for } x \in U, \quad i = 1, 2$$

with some sets $V_1, V_2 \in \mathcal{B}(Y)$. Then we have $(\phi(\alpha_0) \neq 0)$ that

$$F_i(\alpha_0^n x) = \phi(\alpha_0)^n F_i(x) \quad \text{for } \alpha \in A, \quad x \in U, \quad n \in \mathbb{N}, \quad i = 1, 2.$$

Hence for arbitrary $x \in U$ we get

$$\begin{aligned} F_1(x) - F_2(x) &= \phi(\alpha_0)^{-n}F_1(\alpha_0^n x) - \phi(\alpha_0)^{-n}F_2(\alpha_0^n x) \\ &= \phi(\alpha_0)^{-n}F_1(\alpha_0^n x) - \phi(\alpha_0)^{-n}f(\alpha_0^n x) \\ &\quad + \phi(\alpha_0)^{-n}f(\alpha_0^n x) - \phi(\alpha_0)^{-n}F_2(\alpha_0^n x) \\ &\in \phi(\alpha_0)^{-n}K(\alpha_0^n x)V_1 - \phi(\alpha_0)^{-n}K(\alpha_0^n x)V_2 \\ &\subset \frac{1}{|\phi(\alpha_0)^n|} |K(\alpha_0^n x)| \text{aconv } V_1 \\ &\quad + \frac{1}{|\phi(\alpha_0)^n|} |K(\alpha_0^n x)| \text{aconv } V_2 \\ &\subset \left(\frac{\psi(\alpha_0)}{|\phi(\alpha_0)|} \right)^n |K(x)| (\text{aconv } V_1 + \text{aconv } V_2) \end{aligned}$$

for every $n \in \mathbb{N}$. Thus, since $\text{aconv } V_1 + \text{aconv } V_2$ is bounded (cf. Lemma 4(i), (ii)), in virtue of Lemma 4(vi) we obtain $F_1(x) - F_2(x) = 0$ for $x \in U$.

Since $\alpha_0 \in A_1$ was arbitrarily fixed, we derive from (13)

$$F(x) - f(x) \in K(x)V_0 \quad \text{for } x \in U,$$

where

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y). \quad \square$$

The following simply example shows that the estimation obtained in Theorem 4 is the best one.

Example 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x + 2 & \text{for } x \in \langle 2 \rangle, \\ x - 2 & \text{for } x \in -\langle 2 \rangle, \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle. \end{cases}$$

If we consider suitable cases one can check that the assumptions of Theorem 4 are fulfilled with $A = \{-2, 2\}$, $U = \mathbb{R}$, $\phi = \text{id}_A$, $K = 1$, $\psi = 1$, $V(\alpha) = [-2, 2]$ for $\alpha \in \{-2, 2\}$. Next one can verify that the function $F: \mathbb{R} \rightarrow \mathbb{R}$,

$$F(x) = \begin{cases} x & \text{for } x \in \langle \{-2, 2\} \rangle, \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle \end{cases}$$

satisfies

$$F(\alpha x) = \alpha F(x) \quad \text{for } \alpha \in \langle \{-2, 2\} \rangle \cup \{0\}, \quad x \in \mathbb{R},$$

and moreover

$$F(x) - f(x) = \begin{cases} -2 & \text{for } x \in \langle 2 \rangle, \\ 2 & \text{for } x \in -\langle 2 \rangle, \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle. \end{cases}$$

On the other hand $A_1 = \{-2, 2\}$, so

$$V_0 = \bigcap_{\alpha \in A_1} \frac{1}{|\alpha| - 1} \text{seq cl aconv } [-2, 2] = [-2, 2].$$

Since

$$F(x) - f(x) \in \{-2, 0, 2\},$$

the estimation obtained in Theorem 4 is the best one.

As a corollary, from Theorem 4 and Theorem 1 we obtain the following result.

Corollary 1. Assume that the functions $\phi: A \rightarrow \mathbb{K}$ and $f: U \rightarrow Y$ satisfy condition (10). Let $A_1 := \{\alpha \in Z(G) \cap A: |\phi(\alpha)| > \psi(\alpha)\} \neq \emptyset$ and assume that f is K -unbounded.

Then there exists a unique homomorphism $\tilde{\phi}: \langle A \rangle_s \rightarrow \mathbb{K}$ and there exists a unique function $F: U \rightarrow Y$ such that $\tilde{\phi}|_A = \phi$,

$$F(\alpha x) = \tilde{\phi}(\alpha)F(x) \quad \text{for} \quad \alpha \in \langle A \rangle_s, \quad x \in U, \quad (14)$$

and

$$F(x) - f(x) \in K(x)V_0 \quad \text{for} \quad x \in U,$$

where

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y).$$

Proof. From Theorem 4 we derive the existence of a unique function $F: U \rightarrow Y$ and the existence of the bounded set

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y),$$

such that

$$F(\alpha x) = \phi(\alpha)F(x) \quad \text{for} \quad \alpha \in A, \quad x \in U, \quad (15)$$

and

$$F(x) - f(x) \in K(x)V_0, \quad x \in U. \quad (16)$$

Suppose that $F = 0$. Then from (16) we get $f(x) \in K(x) \text{ aconv } V_0$ for every $x \in U$, which is in contradiction to the assumption that f is K -unbounded. Thus $F \neq 0$. It follows from Theorem 1 for Eq. (15) that there exists a unique homomorphism $\tilde{\phi}: \langle A \rangle_s \rightarrow \mathbb{K}$, $\tilde{\phi}|_A = \phi$ such that the function F is $\tilde{\phi}$ -homogeneous, i.e. that Eq. (14) holds. \square

Remark 1. Note that in Corollary 1 the assumption that f is K -unbounded is essential. Indeed, if there exists a set $W \in \mathcal{B}(Y)$ such that $f(x) \in K(x)W$ for $x \in U$ then

$$0 - f(x) \in K(x) \text{ aconv } W \quad \text{for} \quad x \in U.$$

Clearly the function $F_1: U \rightarrow Y$, $F_1(x) = \theta$ is a ϕ -homogeneous one.

On the other hand, in the proof of Theorem 4 we have shown the existence of a unique ϕ -homogeneous function F regardless of the estimation of the difference $F - f$. Thus $F = F_1 = 0$. It is rather difficult to prove any property of such a function ϕ in this case (let us recall that in the case when $F \neq 0$, there exists a unique homomorphism $\tilde{\phi}$ such that $\tilde{\phi}|_{A_1} = \phi$).

4.2. The Group Case

From now on we will assume that $(G, \cdot, 0)$ is a group with zero and that (X, G) is a G -space. Then directly from Theorem 4 and Theorem 3 we deduce the following corollary (we will omit the obvious proof).

Corollary 2. *Assume that $(G, \cdot, 0)$ is an Abelian group, $A \subset G$, $A^* \neq \emptyset$ and let the functions $\phi: A \rightarrow \mathbb{K}$, $\phi(A^*) \subset \mathbb{K}^*$ and $f: U \rightarrow Y$ satisfy (10). Let $A_1 := \{\alpha \in A: |\phi(\alpha)| > \psi(\alpha)\} \neq \emptyset$ and assume that f is a K -unbounded function.*

Then there exists a unique homomorphism $\tilde{\phi}: \langle A \rangle \rightarrow \mathbb{K}$ and a unique function $\bar{F}: \langle A \rangle U \rightarrow Y$ such that $\tilde{\phi}|_A = \phi$, $\bar{F}|_U = F$,

$$\bar{F}(\alpha x) = \tilde{\phi}(\alpha)\bar{F}(x) \quad \text{for } \alpha \in \langle A \rangle, \quad x \in \langle A \rangle U,$$

and

$$\bar{F}(x) - f(x) \in K(x)V_0 \quad \text{for } x \in U,$$

where

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y).$$

In previous theorems we have assumed that the set A_1 is nonempty. It appears that now, when $(G, \cdot, 0)$ is a group, we can consider a weaker condition, that is we may take \neq in a place of $>$. But we must assume that $A \subset G$ and $U \subset X$ are sets such that $A^* \neq \emptyset$, $\langle A \rangle U \subset U$, $\psi: \langle A \rangle \rightarrow [0, \infty)$ is a nonzero homomorphism and the function K satisfies the inequality

$$|K(\alpha x)| \leq \psi(\alpha)|K(x)| \quad \text{for } \alpha \in \langle A \rangle, \quad x \in U. \quad (17)$$

We prove the following theorem, where we use the convention that intersections with an empty index set should be equal to Y .

Theorem 5. *Assume that $\phi: A \rightarrow \mathbb{K}$, $\phi(A^*) \subset \mathbb{K}^*$ and $f: U \rightarrow Y$ satisfy the condition*

$$f(\alpha x) - \phi(\alpha)f(x) \in K(x)V(\alpha) \quad \text{for } \alpha \in A, \quad x \in U. \quad (18)$$

Put $A_1 := \{\alpha \in \mathbf{Z}(G) \cap A^: |\phi(\alpha)| > \psi(\alpha)\}$ and $A_2 := \{\alpha \in \mathbf{Z}(G) \cap A^*: |\phi(\alpha)| < \psi(\alpha)\}$.*

If $A_1 \cup A_2 \neq \emptyset$ then there exists a unique function $F: U \rightarrow Y$ such that

$$F(\alpha x) = \phi(\alpha)F(x) \quad \text{for } \alpha \in \tilde{A}, \quad x \in U \quad (19)$$

and

$$F(x) - f(x) \in K(x)V_0, \quad x \in U,$$

where $\tilde{A} = A$ provided $A_1 \neq \emptyset$ and $\tilde{A} = A^*$ in the case when $A_1 = \emptyset$, and

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \\ \cap \bigcap_{\alpha \in A_2} \left\{ \frac{1}{\psi(\alpha) - |\phi(\alpha)|} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y).$$

Proof. Assume that $A_1 \neq \emptyset$. Then by Theorem 4 there exists a unique function $F_1: U \rightarrow Y$ such that

$$F_1(\alpha x) = \phi(\alpha)F_1(x) \quad \text{for} \quad \alpha \in A, \quad x \in U \quad (20)$$

and

$$F_1(x) - f(x) \in K(x)V_1, \quad x \in U, \quad (21)$$

where

$$V_1 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\}.$$

Now let $A_2 \neq \emptyset$ and fix $\alpha \in A^*$ and $x \in U$. Put $\alpha^{-1}x$ in the place of x in (18). Then from Lemma 4(iv), (v) we get

$$f(x) - \phi(\alpha)f(\alpha^{-1}x) \in K(\alpha^{-1}x)V(\alpha) \subset |K(\alpha^{-1}x)| \text{ aconv } V(\alpha) \\ \subset \psi(\alpha)^{-1}|K(x)| \text{ aconv } V(\alpha) \\ = \psi(\alpha)^{-1}K(x) \text{ aconv } V(\alpha).$$

Thus, since $\alpha \in A^*$ and $x \in U$ were arbitrarily fixed

$$f(\alpha^{-1}x) - \phi(\alpha)^{-1}f(x) \in (\phi(\alpha)\psi(\alpha))^{-1}K(x) \text{ aconv } V(\alpha), \\ \alpha \in A^*, \quad x \in U,$$

and hence

$$f(\alpha x) - \phi(\alpha^{-1})^{-1}f(x) \in (|\phi(\alpha^{-1})|\psi(\alpha^{-1}))^{-1}K(x) \text{ aconv } V(\alpha^{-1}) \\ \text{for} \quad \alpha \in (A^*)^{-1}, \quad x \in U.$$

Put

$$\gamma(\alpha) := \phi(\alpha^{-1})^{-1} \quad \text{and} \quad \tilde{V}(\alpha) := |\phi(\alpha^{-1})|^{-1}\psi(\alpha) \text{ aconv } V(\alpha^{-1})$$

for $\alpha \in (A^*)^{-1}$. Then we get

$$f(\alpha x) - \gamma(\alpha)f(x) \in K(x)\tilde{V}(\alpha) \quad \text{for } \alpha \in (A^*)^{-1}, x \in U. \quad (22)$$

Note that $(A^*)^{-1}U \subset U$. We have

$$\begin{aligned} & \{\alpha \in Z(G) \cap (A^*)^{-1} : |\gamma(\alpha)| > \psi(\alpha)\} \\ &= \{\alpha \in Z(G) \cap A^* : |\gamma(\alpha^{-1})| > \psi(\alpha^{-1})\}^{-1} \\ &= \{\alpha \in Z(G) \cap A^* : |\phi(\alpha)|^{-1} > \psi(\alpha)^{-1}\}^{-1} \\ &= \{\alpha \in Z(G) \cap A^* : |\phi(\alpha)| < \psi(\alpha)\}^{-1} = A_2^{-1} \neq \emptyset. \end{aligned}$$

If we apply Theorem 4 to the condition (22) then we obtain the existence of a unique function $F_2: U \rightarrow Y$ such that

$$F_2(\alpha x) = \gamma(\alpha)F_2(x) \quad \text{for } \alpha \in (A^*)^{-1}, x \in U$$

and

$$F_2(x) - f(x) \in K(x)V_2, \quad x \in U,$$

where

$$V_2 := \bigcap_{\alpha \in (A_2)^{-1}} \left\{ \frac{1}{|\gamma(\alpha)| - \psi(\alpha)} \text{seq cl aconv } \tilde{V}(\alpha) \right\}.$$

Using our notation we get

$$F_2(\alpha x) = \phi(\alpha^{-1})^{-1}F_2(x) \quad \text{for } \alpha \in (A^*)^{-1}, x \in U \quad (23)$$

and

$$F_2(x) - f(x) \in K(x)V_2, \quad x \in U, \quad (24)$$

where

$$\begin{aligned} V_2 &:= \bigcap_{\alpha \in (A_2)^{-1}} \left\{ \frac{1}{|\phi(\alpha^{-1})|^{-1} - \psi(\alpha)} |\phi(\alpha^{-1})|^{-1} \psi(\alpha) \text{seq cl aconv } V(\alpha) \right\} \\ &= \bigcap_{\alpha \in A_2} \left\{ \frac{1}{\psi(\alpha) - |\phi(\alpha)|} \text{seq cl aconv } V(\alpha) \right\}. \end{aligned}$$

Then, from (23) we obtain

$$F_2(x) = \phi(\alpha)F_2(\alpha^{-1}x) \quad \text{for } \alpha \in A^*, x \in U$$

and then, if we put αx in the place of x we get

$$F_2(\alpha x) = \phi(\alpha)F_2(x) \quad \text{for } \alpha \in A^*, x \in U. \quad (25)$$

We are going to show that $F_1 = F_2$ (provided $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$). From (20) and (25) we have

$$F_i(\alpha^n x) = \phi(\alpha)^n F_1(x) \quad \text{for } \alpha \in A^*, x \in U, n \in \mathbb{Z}, i = 1, 2. \quad (26)$$

Fix $x \in U$ and $\alpha \in A_1, \beta \in A_2$ (then we have $\phi(\alpha), \phi(\beta) \neq 0$). From (26), (21) and (24), for every $n \in \mathbb{N}$ we obtain

$$\begin{aligned} F_2(x) - F_1(x) &= \phi(\alpha)^{-n} F_2(\alpha^n x) - \phi(\beta)^n F_1(\beta^{-n} x) \\ &= \left(\frac{\phi(\beta)}{\phi(\alpha)} \right)^n [\phi(\beta)^{-n} F_2(\alpha^n x) - \phi(\alpha)^n F_1(\beta^{-n} x)] \\ &= \left(\frac{\phi(\beta)}{\phi(\alpha)} \right)^n [F_2(\alpha^n \beta^{-n} x) - f(\alpha^n \beta^{-n} x) \\ &\quad + f(\alpha^n \beta^{-n} x) - F_1(\alpha^n \beta^{-n} x)] \\ &\in \left(\frac{|\phi(\beta)|}{|\psi(\beta)|} \frac{|\psi(\alpha)|}{|\phi(\alpha)|} \right)^n K(x) (\text{aconv } V_2 + \text{aconv } V_1). \end{aligned}$$

Since $|\phi(\beta)|/|\psi(\beta)|, \psi(\alpha)/|\phi(\alpha)| < 1$, and with n tending to infinity, and also using the fact that the set $\text{aconv } V_2 + \text{aconv } V_1$ is bounded (cf. Lemma 4(i), (ii)), we get from Lemma 4(vi) $F_1(x) = F_2(x)$.

Put $F = F_1 = F_2$. From (20) and (25) we then obtain (19). Moreover, using (21) and (24) we get

$$F(x) - f(x) \in K(x) (V_1 \cap V_2).$$

Clearly $V_1 \cap V_2 \in \mathcal{B}(Y)$. The uniqueness of F follows from the fact that F_1 and F_2 were the unique functions satisfying (20), (21), and (24), (25), respectively, while the function F satisfies all these conditions. \square

From Theorems 5 and 3 one can derive the following corollary.

Corollary 3. *Let $(G, \cdot, 0)$ be an Abelian group with zero and assume that the functions $\phi: A \rightarrow \mathbb{K}, \phi(A^*) \subset H^*$ and $f: U \rightarrow Y$ satisfy (18). Let $A_1 = \{\alpha \in A^*: |\phi(\alpha)| > \psi(\alpha)\}$ and $A_2 := \{\alpha \in A^*: |\phi(\alpha)| < \psi(\alpha)\}$.*

If $A_1 \cup A_2 \neq \emptyset$ and f is a K -unbounded function then there exists a unique homomorphism $\tilde{\phi}: \langle \tilde{A} \rangle \rightarrow \mathbb{K}$ and a unique function $\tilde{F}: \langle \tilde{A} \rangle U \rightarrow Y$ such that $\tilde{\phi}|_{\tilde{A}} = \phi|_{\tilde{A}}, \tilde{F}|_U = F$,

$$F(\alpha x) = \tilde{\phi}(\alpha) F(x) \quad \text{for } \alpha \in \langle \tilde{A} \rangle, x \in U,$$

and

$$\tilde{F}(x) - f(x) \in K(x) V_0 \quad \text{for } x \in U,$$

where $\tilde{A} = A$ provided $A_1 \neq \emptyset$, $\tilde{A} = A^*$ when $A_1 = \emptyset$, and

$$V_0 := \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\phi(\alpha)| - \psi(\alpha)} \text{seq cl aconv } V(\alpha) \right\} \\ \cap \bigcap_{\alpha \in A_2} \left\{ \frac{1}{\psi(\alpha) - |\phi(\alpha)|} \text{seq cl aconv } V(\alpha) \right\} \in \mathcal{B}(Y).$$

Remark 2. Note that in Corollaries 1, 2, 3 the assumption that f is K -unbounded is a crucial only in the case when we do not know whether there exists a homomorphism $\tilde{\phi}$ such that $\tilde{\phi}|_A = \phi$. In the case when such a homomorphism exists those corollaries remain true without the assumption on the K -unboundedness of f .

Corollary 4 (cf. [1], [15]). *Let X be a real linear space and let Y be a Banach space. Fix $p \geq 0$ and assume that a function $f: X \rightarrow Y$ satisfies*

$$\|f(\alpha x) - \alpha f(x)\| \leq \varepsilon |\alpha|^p \quad \text{for } \alpha \in \mathbb{R}, \quad x \in X,$$

with some $\varepsilon \geq 0$. Then f is a homogeneous function.

Proof. Put $\phi = \text{id}_{\mathbb{R}}$, $\psi = 1$, $K = 1$, $V(\alpha) = \{y \in Y: \|y\| \leq \varepsilon |\alpha|^p\}$, $A_1 = \{\alpha \in \mathbb{R}: |\alpha| > 1\}$, and $A_2 = \{\alpha \in \mathbb{R}^*: |\alpha| < 1\}$. Then from Corollary 3 and from Remark 2 we obtain that there exists a unique homogeneous function $\bar{F}: X \rightarrow Y$ such that

$$\bar{F}(x) - f(x) \in V_0 \quad \text{for } x \in X,$$

where

$$V_0 = \bigcap_{\alpha \in A_1} \left\{ \frac{1}{|\alpha| - 1} \text{seq cl aconv } V(\alpha) \right\} \\ \cap \bigcap_{\alpha \in A_2} \left\{ \frac{1}{1 - |\alpha|} \text{seq cl aconv } V(\alpha) \right\}.$$

Since for $p = 0$ we have $\inf_{\alpha \in A_1} 1/(|\alpha| - 1) = 0$, whereas for $p > 0$, $\inf_{\alpha \in A_2} |\alpha|^p/(1 - |\alpha|) = 0$, this implies $V_0 = \{0\}$. Thus $f = \bar{F}$. \square

We have also the following

Corollary 5. *Let $(G, \cdot, 0)$ be a group with zero and let (X, G) be a G -space. Assume that Y is a Banach space and let $\delta: G \rightarrow [0, \infty)$ be given. Let a nonzero homomorphism $\phi: A \rightarrow \mathbb{K}$ and a function $f: X \rightarrow Y$ satisfy the condition*

$$\|f(\alpha x) - \phi(\alpha)f(x)\| \leq \delta(\alpha) \quad \text{for } \alpha \in G, \quad x \in X.$$

Put $A_1 := \{\alpha \in Z(G)^*: |\phi(\alpha)| > 1\}$ and $A_2 := \{\alpha \in Z(G)^*: |\phi(\alpha)| < 1\}$.

If $A_1 \cup A_2 \neq \emptyset$ then there exists a unique ϕ -homogeneous function $F: U \rightarrow Y$ such that

$$\|F(x) - f(x)\| \leq C \quad \text{for } x \in U,$$

where

$$C := \min \left(\inf_{\alpha \in A_1} \frac{1}{|\phi(\alpha)| - 1}, \inf_{\alpha \in A_2} \frac{1}{1 - |\phi(\alpha)|} \right).$$

5. Completeness

In [5] it was shown that a normed space Y has to be a Banach space provided that for some Abelian group A containing an element of infinite order and for all functions $f: A \rightarrow Y$ such that the Cauchy difference $f(x + y) - f(x) - f(y)$ is bounded, there is some additive function h such that $f - h$ is bounded. (For a survey on the original stability question, whether for some f as above there is some additive function h , such that $f - h$ is bounded, see, for example, [3].)

The aim of this section is to show a similar result for the ϕ -homogeneity equation, namely we have

Theorem 6. *Let A be a group which is isomorphic to $H \times A'$ with some subgroup H of \mathbb{K} ($\in \{\mathbb{R}, \mathbb{C}\}$) such that H contains some element z_0 of modulus different from 1. Assume that there is an action $\cdot: A \times X \rightarrow X$ of A on some set X and assume that the stabilizer $A_{x_0} := \{\alpha \in A: \alpha x_0 = x_0\}$ is trivial for some $x_0 \in X$. Furthermore let Y be a normed space.*

Assume furthermore that for all functions $f: X \rightarrow Y$ and all homomorphisms $\varphi: A \rightarrow \mathbb{K}^ := \mathbb{R} \setminus \{0\}$ such that for some positive ε, δ*

$$\|f(\alpha x) - \varphi(\alpha)f(x)\| \leq \varepsilon|\varphi(\alpha)| + \delta$$

for all $\alpha \in A$ and $x \in X$ there is some $h: X \rightarrow Y$ being φ -homogeneous – which means that $h(\alpha x) = \varphi(\alpha)h(x)$ for all $\alpha \in A$ and $x \in X$ – such that $f - h$ is bounded.

Then Y is a Banach space, i.e., a complete normed space.

Proof. Let $z_0 \in H$ and (without loss of generality) $|z_0| =: r > 1$. Let $(y_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in Y . Since it is enough to show that this sequence contains a convergent subsequence we may – again without loss of generality – assume that

$$\|y_{n+m} - y_n\| \leq r^{-n} \quad \text{for all } n, m \in \mathbb{N}. \quad (27)$$

Then we define $f: X \rightarrow Y$ in the following way. We put $f|_{Ax} := 0$ if $X \ni x \neq x_0$. ($Ax := A(x) := \{\alpha x \mid \alpha \in A\}$ is the orbit of x with respect to

the action of A on X .) On Ax_0 we define $f(\alpha x_0)$ for $\alpha = (z, \alpha') \in A = H \times A'$ as

$$f((z, \alpha')x_0) := \begin{cases} zy_n & \text{if } |z| \geq 1 \text{ and } r^n \leq |z| < r^{n+1}, \\ 0 & \text{if } 0 < |z| < 1. \end{cases}$$

Then f is well defined since $(z, \alpha')x_0 = (z_1, \alpha'_1)x_0$ implies that $z = z_1$ (and $\alpha' = \alpha'_1$) because of our assumption that $A_{x_0} = \{1\}$.

The function φ defined by $\varphi((z, \alpha')) := z$ obviously is a homomorphism from A to \mathbb{K}^* . We want to show that for suitable $\varepsilon, \delta > 0$

$$\|f((z, \alpha')x) - zf(x)\| \leq |z|\varepsilon + \delta \quad (28)$$

for all $(z, \alpha') \in A$ and all $x \in X$.

This is obvious if $x \notin Ax_0$ because in this case the left-hand side of (28) vanishes. So let us assume that $x = (z, \alpha')x_0 \in Ax_0$ and take any $(z_1, \alpha'_1) \in A$. We have to consider several cases.

Case 1. Let $|z|, |z_1| \geq 1$. Then there are nonnegative integers m and n such that $r^n \leq |z| < r^{n+1}$ and $r^m \leq |z_1| < r^{m+1}$, implying $r^{n+m} \leq |zz_1| < r^{n+m+2}$. Accordingly

$$f((z_1, \alpha'_1)x) = f((z_1, \alpha'_1)(z, \alpha')x_0) = f((zz_1, \alpha'\alpha'_1)x_0) = zz_1y_{n+m+\sigma}$$

with some $\sigma \in \{0, 1\}$ and

$$\begin{aligned} f((z_1, \alpha'_1)x) - \varphi((z_1, \alpha'_1))f(x) &= zz_1y_{n+m+\sigma} - z_1zy_n \\ &= zz_1(y_{n+m+\sigma} - y_n). \end{aligned}$$

Thus

$$\begin{aligned} \|f((z_1, \alpha'_1)x) - \varphi((z_1, \alpha'_1))f(x)\| &= |z||z_1|\|y_{n+m+\sigma} - y_n\| \\ &\leq r^{n+1}|z_1|r^{-n} = |z_1|r. \end{aligned}$$

Case 2. If $|z| \geq 1$ and $|z_1| < 1$ we have to divide the consideration into two subclasses.

(a) If $|zz_1| < 1$ we have $f((z_1, \alpha'_1)x) = 0$ and $f(x) = zy_n$, implying

$$\|f((z_1, \alpha'_1)x) - z_1f(x)\| = |z_1||z|\|y_n\| \leq M := \sup_{k \in \mathbb{N}} \|y_k\|.$$

(b) If $|zz_1| \geq 1$ we have $r^m \leq |zz_1| < r^{m+1}$ for some integer m with $0 \leq m \leq n$. Accordingly $f((z_1, \alpha'_1)x) - \varphi((z_1, \alpha'_1))f(x) = zz_1(y_m - y_n)$ and

$$\|f((z_1, \alpha'_1)x) - \varphi((z_1, \alpha'_1))f(x)\| = |zz_1|\|y_p - y_n\| \leq r^{m+1}r^{-m} = r.$$

Case 3. In the case $|z| < 1$, $|z_1| \geq 1$ we again have two subcases.

(a) $|zz_1| < 1$ implies the desired estimate since then $f((z_1, \alpha'_1)x) = f(x) = 0$.

(b) If $|zz_1| \geq 1$ let $m \in \mathbb{N}_0$ be such that $r^m \leq |z_1| < r^{m+1}$. Then $r^n \leq |zz_1| < r^{n+1}$ for some $0 \leq n \leq m$. Thus

$$\|f((z_1, \alpha'_1)x) - \varphi((z_1, \alpha'_1))f(x)\| = |zz_1| \|y_n - y_m\| \leq r.$$

Case 4. The last case $|z|, |z_1| < 1$ is trivial since here as in the first subcase of the previous case all interesting terms vanish.

If we put $\varepsilon := r$ and $\delta := \max\{M, r\}$ we see that the hypotheses of the theorem are satisfied. Thus there is some φ -homogeneous $h: X \rightarrow Y$ and some constant N such that $\|f(x) - h(x)\| \leq N$ for all $x \in X$. Putting $x_n := (z_0, 1)^n x_0$ we have $f(x_n) = z_0^n y_n$ and $h(x_n) = z_0^n h(x_0)$. This implies

$$\|f(x_n) - h(x_n)\| = |z_0|^n \|y_n - h(x_0)\| \leq N.$$

Dividing by $r^n = |z_0|^n$ and letting n tending to infinity then shows that y_n tends to $h(x_0) \in Y$, the desired result. \square

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