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Remarks on the Stability of a Functional Equation of Quadratic Type

By

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Abstract

Our aim is to present some *generalized stability results of Ulam-Hyers type* for λ-quadratic functional equations of the form $Q_\lambda(F) = 0$, where $\lambda \in \{1, 2\}$, $Q_\lambda(F)$ is given by

$$Q_\lambda(F)(u, v) := F(u + v) + F(u + S(v)) + (\lambda - 1)(F(u - v) + F(u - S(v))) - 2^\lambda \left(F(u) + F(v) + F\left(\frac{u + S(u) + v - S(v)}{2}\right) + F\left(\frac{u - S(u) + v + S(v)}{2}\right)\right),$$

and the unknown function F is defined on linear spaces $Z = X_1 \times X_2$ and $S = S_{X_1} := P_{X_1} - P_{X_2}$.

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1. Introduction

Different methods to obtain stability properties for functional equations are known. The *direct method* revealed by HYERS in [17], where the Ulam's problem concerning the stability of homomorphisms was affirmatively answered for Banach spaces, arrived at a very large extent and successful use (see, e.g., [1], [3], [32], [16], [22]).

The interested reader may consult [13], [18], [10], [11] and [19] for details.

On the other hand, in [27], [5] and [6] a *fixed point method* was proposed, by showing that many theorems concerning the stability of Cauchy and Jensen equations are consequences of the fixed point alternative. Subsequently, the method has been successfully used, e.g., in [7], [8], [31], [21], [20] or [24]. It is worth noting that the fixed point method introduces a metrical context and better clarifies the ideas of stability, which is seen to be unambiguously related to fixed points of concrete contractive-type operators on suitable (function) spaces.

We present some *generalized Ulam-Hyers stability results* for functional equations of λ-quadratic type. By using both the direct method and the fixed point method, we slightly extend the results in [25], [26], [9], [16], [22], [28], [29] and [30].

2. Functional Equations of λ-Quadratic Type

Let X_1, X_2 and Y be real linear spaces and consider the Cartesian product $Z := X_1 \times X_2$ together with the linear selfmappings P_{X_1}, P_{X_2} and S, where $P_{X_1}(u) = (u_1, 0)$, $P_{X_2}(u) = (0, u_2)$, $\forall u = (u_1, u_2) \in Z$, and $S = S_{X_1} := P_{X_1} - P_{X_2}$. A function $F: Z \to Y$ is called a *λ-quadratic mapping* $(\lambda \in \{1,2\})$ iff it satisfies, for all $u, v \in Z$, the following equation:

$$Q_\lambda(F)(u,v) := F(u+v) + F(u+S(v)) + (\lambda-1)(F(u-v)+F(u-S(v))) - 2^\lambda\left(F(u)+F(v)+F\left(\frac{u+S(u)+v-S(v)}{2}\right) + F\left(\frac{u-S(u)+v+S(v)}{2}\right)\right) = 0. \tag{2.1}$$

Notice that, whenever Z is an inner product space, $F(u) = a \cdot \|P_{X_1}u\|^\lambda \cdot \|P_{X_2}u\|^2$, $\lambda \in \{1,2\}$, defines a solution of (2.1) for each $a \in \mathbb{R}$.

For $\lambda = 1$ a solution $F: Z \to Y$ is called an *Add Q-type mapping*. If F is a solution of (2.1) for $X_1 = X_2 = X$, then $X \times X \ni u = (x,z) \to f(x,z) := F(u) \in Y$ is an *additive-quadratic mapping on X*, i.e., it verifies the following equation [26]:

$$f(x+y, z+w) + f(x+y, z-w) = 2(f(x,z) + f(y,w) + f(x,w) + f(y,z)), \qquad \forall x,y,z,w \in X. \tag{2.2}$$

For $\lambda = 2$, a solution $F\colon Z \to Y$ is called a *Bi Q-type mapping*. If F verifies (2.1) for $X_1 = X_2 = X$, then $u = (x,z) \to f(x,z) := F(u)$ is *a bi-quadratic mapping*, verifying the following equation [25]:

$$f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)$$
$$= 4(f(x,z)+f(y,w)+f(x,w)+f(y,z)), \qquad \forall x,y,z,w \in X. \tag{2.3}$$

Remark 2.1. Any solution F of (2.1) has the following properties:

(i) $F(0) = 0$; F is an odd mapping for $\lambda = 1$ and an even mapping for $\lambda = 2$;

(ii) $F(2^n \cdot u) = 2^{(\lambda+2)n} \cdot F(u),\ \forall u \in Z,\ \forall n \in \mathbb{N}$;

(iii) $F \circ S = F$ and $F \circ P_{X_1} = F \circ P_{X_2} = 0$;

(iv) moreover, if $f(x,z) = F(u)$, where $u = (x,z)$, then

(iv.1) for $\lambda = 1, f$ is additive in the first variable and quadratic in the second variable;

(iv.2) for $\lambda = 2, f$ is quadratic in each variable.

We also have the following

Lemma 2.1. *Suppose $F\colon Z \to Y$ is of the form*

$$F(u) = f_2(z)f_1(x), \qquad \forall u = (x,z) \in Z = X_1 \times X_2,$$

with arbitrary nonzero mappings $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to \mathbb{R}$. Then:

(i) *F is 1-quadratic if f_1 is additive and f_2 is quadratic*;

(ii) *f_1 is additive if F is 1-quadratic and f_2 is quadratic*;

(iii) *f_2 is quadratic if F is 1-quadratic and f_1 is additive*;

(iv) *F is 2-quadratic if and only if f_1 and f_2 are quadratic.*

2.1. The Generalized Ulam-Hyers Stability for λ-Quadratic Equations

Let us consider a control mapping $\Phi\colon Z \times Z \to [0,\infty)$ such that, for all $u, v \in Z$,

$$\Psi(u,v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{2^{(\lambda+2)(i+1)}} < \infty,$$

$$\left(\Psi(u,v) := \sum_{i=1}^{\infty} 2^{(\lambda+2)(i-1)} \Phi\left(\frac{u}{2^i}, \frac{v}{2^i}\right) < \infty, \quad \text{respectively}\right) \tag{2.4}$$

and suppose Y is a Banach space.

Theorem 2.2. *Let $F: Z \to Y$ be such a mapping that $F \circ P_{X_1} + (\lambda - 1) F \circ P_{X_2} = 0$ and*

$$\|Q_\lambda(F)(u,v)\|_Y \le \Phi(u,v), \qquad \forall u, v \in Z. \tag{2.5}$$

Then there exists a unique λ-quadratic mapping $B: Z \to Y$, given by

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{2^{(\lambda+2)n}}, \qquad \left(B(u) = \lim_{n\to\infty} 2^{(\lambda+2)n} \cdot F\left(\frac{u}{2^n}\right)\right), \qquad \forall u \in Z,$$

for which

$$\|F(u) - B(u)\|_Y \le \Psi(u,u), \qquad \forall u \in Z. \tag{2.6}$$

Proof. We shall use the Hyers' direct method. Letting $u = v$ in (2.5), we obtain

$$\left\| \frac{F(2u)}{2^{\lambda+2}} - F(u) \right\|_Y \le \frac{\Phi(u,u)}{2^{\lambda+2}}, \qquad \forall u \in Z.$$

In the next step, as usual, one shows that

$$\left\| \frac{F(2^p u)}{2^{(\lambda+2)p}} - \frac{F(2^m u)}{2^{(\lambda+2)^m}} \right\|_Y \le \sum_{i=p}^{m-1} \frac{\Phi(2^i u, 2^i u)}{2^{(\lambda+2)(i+1)}}, \qquad \forall u \in Z, \tag{2.7}$$

for given integers p, m, with $0 \le p < m$. Using (2.4) and (2.7), $\{F(2^n u)/2^{(\lambda+2)n}\}_{n\ge 0}$ is a Cauchy sequence for any $u \in Z$. Since Y is complete, we can define the mapping $B: Z \to Y$,

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{2^{(\lambda+2)n}}, \qquad \forall u \in Z. \tag{2.8}$$

By using (2.7) for $p = 0$ and $m \to \infty$ we obtain the estimation (2.6).

By (2.5), we have

$$\begin{aligned} &\Bigg\| \frac{F(2^n(u+v))}{2^{(\lambda+2)n}} + \frac{F(2^n(u+S(v)))}{2^{(\lambda+2)n}} \\ &\quad + (\lambda-1)\left(\frac{F(2^n(u-v))}{2^{(\lambda+2)n}} + \frac{F(2^n(u-S(v)))}{2^{(\lambda+2)n}} \right) \\ &\quad - 2^\lambda \left(\frac{F(2^n(u))}{2^{(\lambda+2)n}} + \frac{F(2^n(v))}{2^{(\lambda+2)n}} + \frac{1}{2^{(\lambda+2)n}} F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right) \right. \\ &\quad \left. + \frac{1}{2^{(\lambda+2)n}} F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\Bigg\|_Y \le \frac{\Phi(2^n u, 2^n v)}{2^{(\lambda+2)n}}, \end{aligned}$$

for all $u, v \in Z$. Using (2.4), (2.8) and letting $n \to \infty$, we immediately see that B is a λ-quadratic mapping.

Let B_1 be a λ-quadratic mapping which satisfies (2.6). Then

$$\|B(u) - B_1(u)\|_Y \leq \left\| \frac{B(2^n u)}{2^{(\lambda+2)n}} - \frac{F(2^n u)}{2^{(\lambda+2)n}} \right\|_Y + \left\| \frac{F(2^n u)}{2^{(\lambda+2)n}} - \frac{B_1(2^n u)}{2^{(\lambda+2)n}} \right\|_Y$$

$$\leq 2 \cdot \sum_{k=n}^{\infty} \frac{\Phi(2^k u, 2^k u)}{2^{(\lambda+2)(k+1)}} \longrightarrow 0, \qquad \text{for} \quad n \to \infty.$$

Hence the uniqueness claim for B holds true. □

Let us consider a mapping $\varphi: X \times X \times X \times X \to [0, \infty)$ such that $\forall x, y, z, w \in X$,

$$\psi(x, z, y, w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i z, 2^i y, 2^i w)}{2^{(\lambda+2)(i+1)}} < \infty,$$

$$\left(\psi(x, z, y, w) := \sum_{i=1}^{\infty} 2^{(\lambda+2)(i-1)} \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, \frac{w}{2^i} \right) < \infty, \ \text{respectively} \right).$$

As a direct consequence of Theorem 2.2, for $\lambda = 1/\lambda = 2$, we obtain:

Corollary 2.3. *Suppose that X is a real linear space, Y is a real Banach space and let $f: X \times X \to Y$ be a mapping such that*

$$\|f(x+y, z+w) + f(x+y, z-w) + (\lambda - 1)(f(x-y, z+w) + f(x-y, z-w)) - 2^{\lambda}(f(x,z) + f(y,w) + f(x,w) + f(y,z))\|_Y \leq \varphi(x, z, y, w),$$

and let $f(x, 0) + (\lambda - 1) \cdot f(0, z) = 0$, for all $x, y, z, w \in X$. Then there exists a unique additive-quadratic/bi-quadratic mapping $b: X \times X \to Y$, given by

$$b(x,z) = \lim_{n\to\infty} \frac{f(2^n x, 2^n z)}{2^{(\lambda+2)n}}, \qquad \left(b(x,z) = \lim_{n\to\infty} 2^{(\lambda+2)n} \cdot f\left(\frac{x}{2^n}, \frac{z}{2^n} \right) \right), \quad \forall x, z \in X,$$

such that

$$\|f(x,z) - b(x,z)\|_Y \leq \psi(x, z, x, z), \qquad \forall x, z \in X. \qquad (2.9)$$

Proof. Let us consider $X_1 = X_2 = X$, $u, v \in X \times X$, $u = (x, z)$, $v = (y, w)$, $F(u) = f(x, z)$, and $\Phi(u, v) = \varphi(x, z, y, w)$. Since $\Psi(u, v) = \psi(x, z, y, w) < \infty$, then we can apply Theorem 2.2. Clearly, the mapping b, defined by $b(x, z) = B(u)$ is additive-quadratic/bi-quadratic and verifies (2.6). □

For $\lambda = 1$ in the above Corollary, we obtain the stability result in ([26], Theorem 7) and, for $\lambda = 2$, that in ([25], Theorem 7).

2.2. Stability Results of Aoki-Rassias Type

For particular forms of the mapping Φ in (2.4), we can obtain interesting consequences. We identify stability properties with unbounded control conditions invoking sums (AOKI [1]) and products (RASSIAS [28–30]) of powers of norms.

Let X_1, X_2 and Y be real linear spaces. Suppose that $Z := X_1 \times X_2$ is endowed with a norm $\|u\|_Z$ and that Y is a real Banach space.

Corollary 2.4. *Let $F: Z \to Y$ be a mapping such that*

$$\|Q_\lambda(F)(u,v)\|_Y \le \varepsilon(\|u\|_Z^p + \|v\|_Z^q), \qquad \forall u, v \in Z,$$

where $p, q \in [0, \lambda+2)$ or $p, q \in (\lambda+2, \infty)$ and $\varepsilon \ge 0$ are fixed. If $F \circ P_{X_1} = 0$ and $(\lambda - 1)F \circ P_{X_2} = 0$, then there exists a unique λ-quadratic mapping $B: Z \to Y$, such that

$$\|F(u) - B(u)\|_Y \le \frac{\varepsilon}{|2^{\lambda+2} - 2^p|} \cdot \|u\|_Z^p + \frac{\varepsilon}{|2^{\lambda+2} - 2^q|} \cdot \|u\|_Z^q, \qquad \forall u \in Z.$$

Proof. Consider the mapping $\Phi: Z \times Z \to [0, \infty)$, $\Phi(u,v) = \varepsilon(\|u\|_Z^p + \|v\|_Z^q)$, where $p, q \in [0, \lambda+2)$ or $p, q \in (\lambda+2, \infty)$ and $\varepsilon \ge 0$. Then (see (2.4)),

$$\Psi(u,v) = \varepsilon \cdot \frac{\|u\|_Z^p}{|2^{\lambda+2} - 2^p|} + \varepsilon \cdot \frac{\|v\|_Z^q}{|2^{\lambda+2} - 2^q|} < \infty, \qquad \forall u, v \in Z,$$

and the conclusion follows directly from Theorem 2.2. □

Now, suppose that $X_1 = X_2 = X$, where X is a real normed space, and consider the function $X \times X \ni u = (x, z) \to F(u) = f(x, z)$, where f is mapping $X \times X$ into the real Banach space Y. Although the functions of the form $u \to \|u\| := (\|x\|^r + \|z\|^s)^{1/t}$ may not be norms, the above proofs work as well, and we obtain the following stability properties for λ-quadratic equations:

Corollary 2.5. *Let $f: X \times X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) + (\lambda-1)(f(x-y, z+w) \\ &\qquad + f(x-y, z-w)) - 2^\lambda(f(x,z) + f(y,w) + f(x,w) + f(y,z))\|_Y \\ &\quad \le \varepsilon(\|x\|_X^p + \|y\|_X^p + \|z\|_X^q + \|w\|_X^q), \end{aligned}$$

for all $x, y, z, w \in X$ and for some fixed ε, p, q, with $p, q \in [0, 2+\lambda)$ or $p, q \in (\lambda+2, \infty)$ and $\varepsilon \ge 0$. If $f(x, 0) = 0$ and $(\lambda - 1)f(0, y) = 0$, *for all* $x, y \in X$, *then there exists a unique additive-quadratic/bi-quadratic mapping* $b: X \times X \to Y$, *such that*

$$\|f(x,z) - b(x,z)\|_Y \le \frac{2\varepsilon}{|2^{\lambda+2} - 2^p|} \cdot \|x\|_X^p + \frac{2\varepsilon}{|2^{\lambda+2} - 2^q|} \cdot \|z\|_X^q, \quad \forall x, z \in X$$

for all $x, z \in X$.

Furthermore, by using the means inequality or directly, two interesting results of RASSIAS type can be obtained for products:

Corollary 2.6. *Let* $F: Z \to Y$ *be a mapping such that*

$$\|Q_\lambda(F)(u,v)\|_Y \le \varepsilon \cdot \|u\|_Z^p \cdot \|v\|_Z^q, \qquad \forall u, v \in Z,$$

where $\varepsilon, p, q \ge 0$ *are fixed and* $p + q \ne \lambda + 2$. *If* $F \circ P_{X_1} = 0$ *and* $(\lambda - 1)F \circ P_{X_2} = 0$, *then there exists a unique* λ*-quadratic mapping* $B: Z \to Y$, *such that*

$$\|F(u) - B(u)\|_Y \le \frac{\varepsilon}{|2^{\lambda+2} - 2^{p+q}|} \cdot \|u\|_Z^{p+q}, \qquad \forall u \in Z.$$

Proof. Consider the mapping $\Phi: Z \times Z \to [0, \infty)$, $\Phi(u,v) = \varepsilon \cdot \|u\|_Z^p \cdot \|v\|_Z^q$, where $\varepsilon, p, q \ge 0$ are fixed and $p + q \ne \lambda + 2$. Then (see (2.4))

$$\Psi(u,v) = \varepsilon \cdot \frac{\|u\|_Z^p \cdot \|v\|_Z^q}{|2^{\lambda+2} - 2^{p+q}|} < \infty, \qquad \forall u, v \in Z,$$

so that we can apply Theorem 2.2. □

Corollary 2.7. *Let* $f: X \times X \to Y$ *be a mapping such that*

$$\begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) + (\lambda - 1)(f(x-y, z+w) \\ &\qquad + f(x-y, z-w)) - 2^\lambda(f(x,z) + f(y,w) + f(x,w) + f(y,z))\|_Y \\ &\quad \le \varepsilon \cdot (\|x\|_X^p + \|z\|_X^p) \cdot (\|y\|_X^q + \|w\|_X^q), \end{aligned}$$

for all $x, y, z, w \in X$ *and for some fixed* $\varepsilon, p, q \ge 0$, *with* $p + q \ne \lambda + 2$. *If* $f(x, 0) = 0$ *and* $(\lambda - 1)f(0, y) = 0$, *for all* $x \in X$, *then there exists a unique additive-quadratic/bi-quadratic mapping* $b: X \times X \to Y$, *such that*

$$\|f(x,z) - b(x,z)\|_Y \le \frac{\varepsilon}{|2^{\lambda+2} - 2^{p+q}|} \cdot (\|x\|_X^p + \|z\|_X^p) \cdot (\|x\|_X^q + \|z\|_X^q),$$
$$\forall x, z \in X.$$

2.3. Applications to Additive Equations and to Quadratic Equations

A function $h: X \to Y$, between linear spaces, is called *a mapping of* λ*-order*, $\lambda \in \{1,2\}$, if it satisfies the following equation:

$$h(x+y)+(\lambda-1)h(x-y)=2^{\lambda-1}(h(x)+h(y)), \quad \forall x,y \in X. \quad (2.10)_\lambda$$

Obviously, a mapping of 1-order is an additive mapping and a mapping of 2-order is a quadratic mapping.

For the sake of convenience, we recall the following generalized Ulam-Hyers stability properties of the additive and quadratic functional equations. Let X be a real normed vector space, Y a real Banach space and $\bar{\varphi}: X \times X \to [0,\infty)$ a given mapping.

A_1 ([16], Theorem; see also [12]): *If $\bar{\varphi}$ verifies the condition*

$$\bar{\phi}_1(x,y) := \sum_{i=0}^{\infty} \frac{\bar{\varphi}(2^i x, 2^i y)}{2^{i+1}} < \infty, \qquad \textit{for all} \quad x,y \in X \qquad (2.11)_1$$

and the mapping $\bar{f}: X \to Y$ satisfies the relation

$$\|\bar{f}(x+y)-\bar{f}(x)-\bar{f}(y)\|_Y \le \bar{\varphi}(x,y), \quad \textit{for all} \ \ x,y \in X, \quad (2.12)_1$$

then there exists a unique additive mapping $\bar{a}_1: X \to Y$ which satisfies the inequality

$$\|\bar{f}(x)-\bar{a}_1(x)\|_Y \le \bar{\phi}_1(x,x), \qquad \textit{for all} \quad x \in X. \qquad (2.13)_1$$

A_2 ([22], Theorem 2.2): *If $\bar{\varphi}$ verifies the condition*

$$\bar{\phi}_2(x,y) := \sum_{i=0}^{\infty} \frac{\bar{\varphi}(2^i x, 2^i y)}{2^{2(i+1)}} < \infty, \qquad \textit{for all} \quad x,y \in X \qquad (2.11)_2$$

and the mapping $\bar{f}$: $X \to Y$, with $\bar{f}(0)=0$, satisfies the relation

$$\|\bar{f}(x+y)+\bar{f}(x-y)-2\bar{f}(x)-2\bar{f}(y)\|_Y \le \bar{\varphi}(x,y), \\ \textit{for all} \quad x,y \in X, \qquad (2.12)_2$$

then there exists a unique quadratic mapping $\bar{a}_2: X \to Y$ which satisfies the inequality

$$\|\bar{f}(x)-\bar{a}_2(x)\|_Y \le \bar{\phi}_2(x,x), \qquad \textit{for all} \quad x \in X. \qquad (2.13)_2$$

As a matter of fact, we can show that the above results are consequences of our Theorem 2.2. Namely, we have

Application 1. The stability of Eq. (2.1) implies the generalized Ulam-Hyers stability of the λ-order equation $(2.10)_\lambda$.

Indeed, let $X, Y, \bar{\varphi}: X \times X \to [0, \infty)$ and $\bar{f}: X \to Y$ be as in $\mathbf{A}_\lambda$, $\lambda \in \{1, 2\}$. We take $X_1 = X$ and consider a linear space X_2 such that there exist a quadratic function $\bar{h}: X_2 \to \mathbb{R}$, with $\bar{h}(0) = 0$ and an element $z_0 \in X_2$, such that $\bar{h}(z_0) \neq 0$. (In inner product spaces such a function is, e.g., $z \to \|z\|^2$.) If we set, for $u = (x, z), v = (y, w) \in X \times X_2$,

$$\Phi(u, v) = \Phi(x, z, y, w) = 2|\bar{h}(z) + \bar{h}(w)| \cdot \bar{\varphi}(x, y)$$

and

$$F(u) = F(x, z) = \bar{h}(z) \cdot \bar{f}(x),$$

then, by using the properties of the quadratic mapping and the relations $(2.11)_\lambda$, for $\lambda \in \{1, 2\}$, we easily get

$$\Psi(u, v) = \frac{1}{2} |\bar{h}(z) + \bar{h}(w)| \sum_{i=0}^{\infty} \frac{\bar{\varphi}(2^i x, 2^i y)}{2^{\lambda(i+1)}} < \infty,$$

for all $u, v \in X \times X_2$. At the same time, by $(2.12)_\lambda$,

$$\begin{aligned} \|Q_\lambda(F)(u, v)\|_Y &= 2|\bar{h}(z) + \bar{h}(w)| \cdot \|\bar{f}(x + y) + (\lambda - 1)\bar{f}(x - y) \\ &\quad - 2^{\lambda-1}(\bar{f}(x) + \bar{f}(y))\|_Y \leq 2|\bar{h}(z) + \bar{h}(w)| \cdot \bar{\varphi}(x, y) \\ &= \Phi(u, v), \qquad \forall u, v \in X \times X_2. \end{aligned}$$

Therefore, by Theorem 2.2, there exists a unique mapping of λ-quadratic type, $B: X \times X_2 \to Y$, such that $\|F(u) - B(u)\|_Y \leq \Psi(u, u)$ and

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{2^{n(\lambda+2)}} = \lim_{n\to\infty} \frac{\bar{h}(2^n z)}{2^{2n}} \cdot \frac{\bar{f}(2^n x)}{2^{\lambda n}} = \lim_{n\to\infty} \bar{h}(z) \cdot \frac{\bar{f}(2^n x)}{2^{\lambda n}},$$
$$\forall u = (x, z) \in X \times X_2.$$

We know that $\bar{h}(z_0) \neq 0$. Therefore the limit

$$\bar{a}_\lambda(x) := \lim_{n\to\infty} \frac{\bar{f}(2^n x)}{2^{\lambda n}}$$

exists for every $x \in X$ and, moreover, $B(u) = \bar{h}(z) \cdot \bar{a}_\lambda(x), \forall u = (x, z) \in Z$. Since $\|\bar{h}(z)\bar{f}(x) - B(u)\|_Y \leq \bar{h}(z) \cdot \bar{\phi}_\lambda(x, x)$, $\forall u = (x, z) \in X \times X_2$, then the estimation $(2.13)_\lambda$ is easily seen to hold.

By Lemma 2.1, $\bar{a}_1$ is additive and $\bar{a}_2$ is quadratic. If a mapping of λ-order $\bar{c}_\lambda$ satisfies $(2.13)_\lambda$, then $(x, z) \to \bar{h}(z)\bar{c}_\lambda(x)$ is of λ-quadratic type (again by Lemma 2.1) and has to coincide with B, that is

$\bar{h}(z)\bar{a}_\lambda(x) = \bar{h}(z)\bar{c}_\lambda(x)$, for all $u = (x, z) \in X \times X_2$. Since $\bar{h}$ is nonzero, then $\bar{a}_\lambda(x) = \bar{c}_\lambda(x)$, for all $x \in X$. Hence $\bar{a}_\lambda$ is unique.

Remark 2.2. As in the proof of Application 1 for an *additive* function $\bar{h}: X_2 \to \mathbb{R}$, we can also show, by using Theorem 2.2, that the stability of Eq. (2.1) for $\lambda = 1$ implies the generalized Ulam-Hyers stability of the quadratic equation $(2.10)_2$.

As very particular cases, we obtain the results in AOKI [1] and RASSIAS [28] for additive equations:

Application 2. Let $\bar{f}: X \to Y$ be a mapping such that

$$\|\bar{f}(x+y) - \bar{f}(x) - \bar{f}(y)\|_Y \leq \varepsilon(\|x\|_X^p + \|y\|_X^p), \qquad \text{for all} \quad x, y \in X,$$

and for any fixed $\varepsilon, p \geq 0$, with $p \neq 1$. If $\bar{f}(0) = 0$, then there exists a unique additive mapping $\bar{a}_1: X \to Y$ which satisfies the estimation

$$\|\bar{f}(x) - \bar{a}_1(x)\|_Y \leq \frac{2\varepsilon}{|2 - 2^p|} \cdot \|x\|_X^p, \qquad \text{for all} \quad x \in X.$$

Indeed, let $\bar{h}: \mathbb{R} \to \mathbb{R}, \bar{h}(z) = z^2$ and $\bar{f}: X \to Y$, where X is a normed space and Y a Banach space. We apply Theorem 2.2 for $\lambda = 1$, $X_1 = X$, $X_2 = \mathbb{R}$, $u, v \in X \times \mathbb{R}$, with $u = (x, z)$, $v = (y, w)$ and the mappings

$$F(u) = F(x, z) = z^2 \cdot \bar{f}(x),$$
$$\Phi(u, v) = \Phi(x, z, y, w) = 2(z^2 + w^2) \cdot \varepsilon(\|x\|_X^p + \|y\|_X^p),$$

to obtain the existence of a unique additive mapping $\bar{a}_1$ and the required estimation.

Application 3. Let $\bar{f}: X \to Y$ be a mapping such that

$$\|\bar{f}(x+y) - \bar{f}(x) - \bar{f}(y)\|_Y \leq \theta\big(\|x\|_X^{p/2} \cdot \|y\|_X^{p/2}\big), \qquad \text{for all} \quad x, y \in X,$$

and for any fixed $\theta, p \geq 0$, with $p < 1$. If $\bar{f}(0) = 0$, then there exists a unique additive mapping $\bar{a}_1: X \to Y$ which satisfies the estimation

$$\|\bar{f}(x) - \bar{a}_1(x)\|_Y \leq \frac{\theta}{2 - 2^p} \cdot \|x\|_X^p, \qquad \text{for all} \quad x \in X.$$

Indeed, one can use *either* the mappings

$$F(u) = F(x, z) = z^2 \cdot \bar{f}(x),$$

and

$$\Phi(u, v) = \Phi(x, z, y, w) = 2(z^2 + w^2) \cdot \theta \cdot \|x\|_X^{p/2} \cdot \|y\|_X^{p/2},$$

or the means inequality:

$$\theta\big(\|x\|_X^{p/2} \cdot \|y\|_X^{p/2}\big) \le \frac{\theta}{2}(\|x\|_X^p + \|y\|_X^p)$$

in the preceding corollary.

In particular, we obtain also a stability property of AOKI type for quadratic equations ([9]):

Application 4. Let $\bar{f}$ be a mapping from a real linear space X into a real Banach space Y, such that

$$\|\bar{f}(z+w) + \bar{f}(z-w) - 2\bar{f}(z) - 2\bar{f}(w)\|_Y \le \varepsilon(\|z\|_X^p + \|w\|_X^p), \quad \text{for all} \quad z, w \in X,$$

and for some fixed $\varepsilon, p \ge 0$, with $p \ne 2$. If $\bar{f}(0) = 0$, then there exists a unique quadratic mapping $\bar{a}_2 : X \to Y$ which satisfies the estimation

$$\|\bar{f}(z) - \bar{a}_2(z)\|_Y \le \frac{2\varepsilon}{|2^2 - 2^p|} \cdot \|z\|_X^p, \qquad \text{for all} \quad z \in X.$$

For the *proof*, let $\bar{h}: \mathbb{R} \to \mathbb{R}, \bar{h}(x) = x$. We apply Theorem 2.2 for $\lambda = 1$, $X_1 = \mathbb{R}$, $X_2 = X$, $u, v \in \mathbb{R} \times X$, with $u = (x, z)$, $v = (y, w)$ and the mappings $F(u) = F(x, z) = x \cdot \bar{f}(z)$, $\Phi(u, v) = \Phi(x, z, y, w) = |x + y| \cdot \varepsilon(\|z\|_X^p + \|w\|_X^p)$, to obtain the existence of a unique quadratic mapping $\bar{a}_2$ and the required estimation.

Similarly, by choosing $F(u) = F(x, z) = x \cdot \bar{f}(z)$ and $\Phi(u, v) = \Phi(x, z, y, w) = |x + y| \cdot \varepsilon \cdot \|z\|_X^p \cdot \|w\|_X^q$, we obtain a stability of RASSIAS type [30]:

Application 5. Let $\bar{f}$ be a mapping from a real linear space X into a real Banach space Y such that $\bar{f}(0) = 0$ and

$$\|\bar{f}(z+w) + \bar{f}(z-w) - 2\bar{f}(z) - 2\bar{f}(w)\|_Y \le \varepsilon \cdot \|z\|_X^p \cdot \|w\|_X^q, \quad \forall z, w \in X,$$

for some fixed $\varepsilon, p, q \ge 0$, with $p + q \ne 2$. Then there exists a unique quadratic mapping $\bar{a}_2 : X \to Y$ which satisfies the estimation

$$\|\bar{f}(z) - \bar{a}_2(z)\|_Y \le \frac{\varepsilon}{|2^2 - 2^{p+q}|} \cdot \|z\|_X^{p+q}, \qquad \forall z \in X.$$

3. A Second Stability Result by the Fixed Point Method

We will show that Corollary 2.4 and Corollary 2.6 can be essentially extended by using *a fixed point method*. The method is seen plainly related to some fixed point of a concrete operator. Specifically, our control conditions are perceived to be responsible for three fundamental facts: Actually, they ensure

1) the *contraction property* of a Schröder type operator J and

2) the first two successive approximations, f and Jf, to be at a *finite distance*.

And, moreover, they force

3) the fixed point function of J to be a *solution of the initial equation*.

Firstly, we prove an auxiliary result of stability for the following equation in a single variable

$$w \circ g \circ \eta = g.$$

Let us consider a Lipschitzian function $w\colon Y \to Y$, with the Lipschitz constant L_w, and the mappings $f\colon G \to Y$, $\eta\colon G \to G$, where G is a nonempty set and Y is a Banach space.

Lemma 3.1. *Suppose that the mapping f satisfies an inequality of the form*

$$\|(w \circ f \circ \eta)(x) - f(x)\|_Y \leq \psi(x), \qquad \forall x \in G, \qquad (C_\psi)$$

where $\psi\colon G \to [0,\infty)$. If there exists $L<1$ such that the mapping ψ has the property

$$L_w \cdot (\psi \circ \eta)(x) \leq L\psi(x), \qquad \forall x \in G, \qquad (H_\psi)$$

then there exists a unique mapping $c\colon G \to Y$,

$$c(x) := \lim_{n\to\infty} (w^n \circ c \circ \eta^n)(x), \qquad \forall x \in G,$$

which satisfies the equation

$$(w \circ c \circ \eta)(x) = c(x), \qquad \forall x \in G$$

and the inequality

$$\|f(x) - c(x)\|_Y \leq \frac{\psi(x)}{1-L}, \qquad \forall x \in G. \qquad (Est_\psi)$$

Proof. Let us consider the set $\mathcal{E} := \{g\colon G \to Y\}$ and introduce a *complete generalized metric* on $\mathcal{E}$ (as usual, $\inf \emptyset = \infty$):

$$d(g,h) = d_\psi(g,h) = \inf\{K \in \mathbb{R}_+, \|g(x) - h(x)\|_Y \leq K\psi(x), \forall x \in G\}. \qquad (GM_\psi)$$

Now, define the mapping

$$J\colon \mathcal{E} \to \mathcal{E}, Jg(x) := (w \circ g \circ \eta)(x). \qquad (OP)$$

Step I. By using the hypothesis (H_ψ), we show that J *is strictly contractive* on $\mathcal{E}$.

We can write, for any $g, h \in \mathcal{E}$:

$$d(g,h) < K \Longrightarrow \|g(x) - h(x)\|_Y \leq K\psi(x), \qquad \forall x \in G.$$

On the other hand,

$$\begin{aligned}\|Jg(x) - Jh(x)\|_Y &= \|w(g(\eta(x))) - w(h(\eta(x)))\|_Y \\ &\leq L_w \cdot \|g(\eta(x)) - h(\eta(x))\|_Y \leq L_w \cdot K \cdot \psi(\eta(x)) \\ &\leq K \cdot L \cdot \psi(x), \quad \forall x \in G \Longrightarrow d(Jg, Jh) \leq LK.\end{aligned}$$

Therefore, we see that

$$d(Jg, Jh) \leq Ld(g,h), \qquad \forall g, h \in \mathcal{E}, \tag{CC_L}$$

that is J is a *strictly contractive* self-mapping of $\mathcal{E}$, with the constant $L < 1$.

Step II. Obviously, $d(f, Jf) < \infty$.

In fact, by using the relation (C_ψ), it results that $d(f, Jf) < 1$.

Step III. We can apply the fixed point alternative (see, e.g., [5]), and we obtain the existence of a mapping $c\colon G \to Y$ such that:

— c is a fixed point of J, that is

$$(w \circ c \circ \eta)(x) = c(x), \qquad \forall x \in G. \tag{3.1}$$

The mapping c is the unique fixed point of J in the set

$$\mathcal{F} = \{g \in \mathcal{E},\ d(f,g) < \infty\}.$$

This says that c is the unique mapping with *both* the properties (3.1) and (3.2), where

$$\exists K \in (0,\infty) \ \text{ such that } \ \|c(x) - f(x)\|_Y \leq K\psi(x), \ \forall x \in G. \tag{3.2}$$

— $d(J^n f, c) \underset{n\to\infty}{\longrightarrow} 0$, which implies the equality

$$c(x) := \lim_{n\to\infty} (w^n \circ c \circ \eta^n)(x), \qquad \forall x \in G. \tag{3.3}$$

— $d(f,c) \leq \dfrac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,c) \leq \frac{1}{1-L},$$

that is (Est_ψ) is seen to be true. □

Let X_1, X_2 be linear spaces, $Z := X_1 \times X_2, Y$ a Banach space, and consider an arbitrary mapping $\Phi\colon Z \times Z \to [0,\infty)$.

Theorem 3.2. *Let* $F\colon Z\to Y$ *be such a mapping for which* $F\circ P_{X_1}+(\lambda-1)F\circ P_{X_2}=0$ *and suppose that*

$$\|Q_\lambda(F)(u,v)\|_Y \le \Phi(u,v), \qquad \forall u,v\in Z. \tag{2.5}$$

If there exists $L<1$ *such that the mapping*

$$u\to\Omega(u)=\Phi\left(\frac{u}{2},\frac{u}{2}\right)$$

verifies the condition

$$\Omega(u)\le L\cdot 2^{\lambda+2}\cdot\Omega\left(\frac{u}{2}\right), \qquad \forall u\in Z, \tag{H_λ}$$

and the mapping Φ *has the property*

$$\lim_{n\to\infty}\frac{\Phi(2^n u,2^n v)}{2^{(\lambda+2)n}}=0, \qquad \forall u,v\in Z, \tag{H_λ^*}$$

then there exists a unique λ*-quadratic mapping* $B\colon Z\to Y$, *such that*

$$\|F(u)-B(u)\|_Y\le\frac{L}{1-L}\Omega(u), \qquad \forall u\in Z. \tag{Est}$$

Proof. If we set $u=v$ in the relation (2.5), then we see that

$$\|F(2u)-2^{\lambda+2}F(u)\|_Y\le\Omega(2u), \qquad \forall u\in Z.$$

Hence

$$\left\|\frac{F(2u)}{2^{\lambda+2}}-F(u)\right\|_Y\le\frac{\Omega(2u)}{2^{\lambda+2}}, \qquad \forall u\in Z. \tag{3.4}$$

Now we can apply Lemma 3.1, with $w,\eta\colon Z\to Y$, $\psi\colon Z\to[0,\infty)$,

$$w(u):=\frac{u}{2^{\lambda+2}}, \qquad \eta(u):=2u, \qquad \psi(u):=\frac{\Omega(2u)}{2^{\lambda+2}}.$$

Clearly, $L_w=1/2^{\lambda+2}$ and, by using (3.4) and the hypothesis (H_λ), we obtain that (C_ψ) and (H_ψ) hold.

Then there exists a unique mapping $B\colon Z\to Y$,

$$B(u):=\lim_{n\to\infty}(w^n\circ B\circ\eta^n)(u)=\lim_{n\to\infty}\frac{F(2^n u)}{2^{(\lambda+2)n}}, \qquad \forall u\in Z, \tag{3.5}$$

which satisfies the following equation

$$(w\circ B\circ\eta)(u)=B(u)\Leftrightarrow B(2u)=2^{\lambda+2}B(u), \qquad \forall u\in Z$$

and the inequality

$$\|F(u)-B(u)\|_Y \le \frac{\psi(u)}{1-L} = \frac{\Omega(2u)}{2^{\lambda+2}}\cdot\frac{1}{1-L} \le \Omega(u)\frac{L}{1-L}, \qquad \forall u\in Z.$$

The statement that B is a λ-quadratic mapping is easily seen: If we replace u by $2^n u$ and v by $2^n v$ in (2.5), then we obtain

$$\begin{aligned}&\Bigg\| \frac{F(2^n(u+v))}{2^{(\lambda+2)n}} + \frac{F(2^n(u+S(v)))}{2^{(\lambda+2)n}} \\ &\quad + (\lambda-1)\left(\frac{F(2^n(u-v))}{2^{(\lambda+2)n}} + \frac{F(2^n(u-S(v)))}{2^{(\lambda+2)n}}\right) - 2^{\lambda}\left(\frac{F(2^n(u))}{2^{(\lambda+2)n}}\right. \\ &\quad + \frac{F(2^n(v))}{2^{(\lambda+2)n}} + \frac{1}{2^{(\lambda+2)n}}F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right) \\ &\quad \left. + \frac{1}{2^{(\lambda+2)n}}F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\Bigg\|_Y \le \frac{\Phi(2^n u, 2^n v)}{2^{(\lambda+2)n}},\end{aligned}$$

for all $u, v\in Z$. By using (3.5) and (H_λ^*) and letting $n\to\infty$, we see that B satisfies (2.1). □

Example 3.1. If we apply Theorem 3.2 with the mappings $\Phi\colon Z\times Z\to [0,\infty)$ given by $(u,v)\to\varepsilon(\|u\|_Z^p+\|v\|_Z^q)$ and $(u,v)\to\varepsilon\|u\|_Z^p\cdot\|v\|_Z^q$, then we obtain the stability results in Corollary 2.4 and Corollary 2.6, respectively.

As it is well known (see [15, 18, 9]), GAJDA/CZERWIK showed that the additive/quadratic equation $(2.12)_\lambda$ is *not stable* for $\bar\varphi(x,y)$ of the form $\varepsilon(\|x\|^\lambda+\|y\|^\lambda)$, ε being a given positive constant ($\lambda\in\{1,2\}$). In fact, it has been proved that there exists a mapping $\bar f_\lambda\colon\mathbb{R}\to\mathbb{R}$ such that $(2.12)_\lambda$ holds with the above $\bar\varphi$, and there exists *no* additive/quadratic mapping $\bar a$ to verify

$$|\bar f_\lambda(x)-\bar a_\lambda(x)|\le c(\varepsilon)|x|^\lambda, \qquad \text{for all} \quad x\in\mathbb{R}.$$

This suggests the following

Example 3.2. Let $X_1=X_2=Y=\mathbb{R}$, with the Euclidean norm, and $\bar h\colon\mathbb{R}\to\mathbb{R}$ a quadratic function with $\bar h(0)=0, \bar h(1)=1$. Then *Eq.* (2.1) *is not stable* for

$$\Phi(u,v)=\Phi(x,z,y,w)=2\varepsilon\cdot(|x|^\lambda+|y|^\lambda)(\bar h(z)+\bar h(w)). \tag{3.6}$$

In fact, we can show that there exists an F for which the relation (2.5) holds and there exists *no Add Q/Bi Q*-type mapping $B\colon X_1\times X_2\to Y$

to verify

$$|F(u) - B(u)| \le c(\varepsilon)\bar{h}(z)|x|^{\lambda}, \qquad \forall u = (x, z) \in X_1 \times X_2. \quad (3.7)$$

Indeed, for $F(u) = F(x, z) = \bar{h}(z) \cdot \bar{f}_{\lambda}(x)$, and Φ as in (3.6), (2.5) holds. Therefore

$$|f(x + y) + (\lambda - 1)f(x - y) - 2^{\lambda-1}(f(x) + h(y))| \le \varepsilon(|x|^{\lambda} + |y|^{\lambda}), \quad \text{for all} \quad x, y \in X_1.$$

Let us suppose, for a contradiction, that there exists an *Add Q/Bi Q*-type mapping B which verifies (3.7). By Remark 2.1, the mapping $\bar{a}_{\lambda}: X_1 \to Y$, $\bar{a}_{\lambda}(x) := B(x, 1)$ is a solution for $(2.10)_{\lambda}$. The estimation (3.7) gives us

$$|\bar{f}_{\lambda}(x) - \bar{a}_{\lambda}(x)| \le c(\varepsilon)|x|^{\lambda}, \qquad \forall x \in X_1,$$

in contradiction with the above result of GAJDA/CZERWIK.

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