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The Invariant Measure for the Two-Dimensional Parry-Daniels Map

By

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Abstract

The Parry-Daniels map T has an exceptional set Γ which can be seen as a strange attractor for T. The density of the invariant measure is given. Some remarks on the exceptional set for the mixture of the Selmer algorithm and the fully subtractive algorithm are added.

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Let $x = (x_0, x_1, x_2) \in (\mathbb{R}^+)^3$ and let π be a permutation of the indices such that $x_{\pi 0} \leq x_{\pi 1} \leq x_{\pi 2}$. The Poincaré map P is defined as

$$P(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - x_{\pi 0}, x_{\pi 2} - x_{\pi 1}).$$

We introduce

$$\Sigma^2 = \{x \in (\mathbb{R}^+)^3 : x_0 + x_1 + x_2 = 1\}.$$

Then the Parry-Daniels map $T\colon \Sigma^2 \to \Sigma^2$ is defined as

$$T(x_0, x_1, x_2) = \left(\frac{x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 1} - x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 2} - x_{\pi 1}}{x_{\pi 2}}\right),$$
$$\pi \in \{\varepsilon, (01), (02), (12), (012), (021)\}.$$

We introduce the notation

$$x^{(k)} = (x_0^{(k)}, x_1^{(k)}, x_2^{(k)}) := P^k x.$$

We define

$$\sigma(x) := \sum_{k\geq 0} \max(x_0^{(k)}, x_1^{(k)}).$$

The following result could be proved (SCHWEIGER [2], NOGUEIRA [1]).

Let

$$\Gamma := \bigcap_{s=0}^{\infty} \bigcup_{\pi_1,\dots,\pi_s \in \{\varepsilon,(01)\}} B(\pi_1,\dots,\pi_s),$$

then $\Gamma = \{x \in \Sigma^2 \colon \sigma(x) \leq x_2\}$ and $\lambda(\Gamma) > 0$. Since T is ergodic with respect to Lebesgue measure, we obtain

$$\Sigma^2 = \bigcup_{j=0}^{\infty} T^{-j}\Gamma.$$

Note that $\sigma(x)$ is convergent for almost all directions $\theta = x_0/x_1$ or $\theta = x_1/x_0$, $0 \leq \theta \leq 1$.

Since on Σ^2 the relation $x_2 = 1 - x_0 - x_1$ holds, we restrict our attention to the first coordinates, i.e. to the domain $\{(x_0, x_1) \colon 0 \leq x_0, 0 \leq x_1, 0 \leq x_0 + x_1 \leq 1\}$.

Theorem. *The function*

$$h(x_0, x_1) = \frac{1}{x_0(x_0 + x_1)(1 - x_0 - x_1 - \sigma(x_0, x_1))}$$

is an invariant density for T restricted to Γ.

Proof. The map T restricted to Γ has only two inverse branches

$$V(\varepsilon)(x_0, x_1) = \left(\frac{x_0}{1 + 2x_0 + x_1}, \frac{x_0 + x_1}{1 + 2x_0 + x_1}\right),$$

$$V(01)(x_0, x_1) = \left(\frac{x_0 + x_1}{1 + 2x_0 + x_1}, \frac{x_0}{1 + 2x_0 + x_1}\right).$$

Then

$$h(V_0(\varepsilon)(x_0,x_1))\omega(\varepsilon,x_0,x_1) + h(V(01)(x_0,x_1))\omega(01;x_0,x_1)$$

$$= \frac{1}{x_0(2x_0 + x_1)\left(1 - (1 + 2x_0 + x_1)\sigma\left(\frac{x_0}{1 + 2x_0 + x_1}, \frac{x_0 + x_1}{1 + 2x_0 + x_1}\right)\right)}$$

$$+\frac{1}{x_0(2x_0+x_1)\left(1-(1+2x_0+x_1)\sigma\left(\frac{x_0}{1+2x_0+x_1},\frac{x_0+x_1}{1+2x_0+x_1}\right)\right)}.$$

We note the following properties of the function σ:

$$\sigma(\lambda y_0,\lambda y_1)=\lambda\sigma(y_0,y_1),$$
$$\sigma(y_0,y_1)=\sigma(y_1,y_0),$$
$$\sigma(x_0,x_0+x_1)=x_0+x_1+\sigma(x_0,x_1).$$

Therefore

$$(1+2x_0+x_1)\sigma\left(\frac{x_0}{1+2x_0+x_1},\frac{x_0+x_1}{1+2x_0+x_1}\right)=x_0+x_1+\sigma(x_0,x_1).$$

Hence

$$h(V(\varepsilon)(x_0,x_1))\omega(\varepsilon;x_0,x_1)+h(V(01)(X_0,x_1))\omega(01;x_0,x_1)=h(x_0,x_1).$$

Remark 1. The set Γ can be described as consisting of all needles emanating from $(0,0)$ which are given by the equations

$$x_0=\lambda,\qquad x_1=\lambda\theta,\qquad 0\le\lambda\le\frac{1}{1+\theta+S(\theta)}$$

or

$$x_0=\lambda\theta,\qquad x_1=\lambda,\qquad 0\le\lambda\le\frac{1}{1+\theta+S(\theta)},$$
$$S(\theta)=\sigma(\theta,1)=\sigma(1,\theta),\qquad 0\le\theta\le 1.$$

Therefore the equation

$$x_0+x_1+\sigma(x_0,x_1)=1$$

can be viewed as referring to the boundary of Γ in some sense (the other parts of the boundary are given by $x_0=0$ and $x_1=1$).

Remark 2. This remark concerns the paper SCHWEIGER [3]. In this paper the Selmer algorithm S and the Fully Subtractive algorithm T were considered. The following theorem was proved:

Theorem. *Let* $\Gamma=(x_1,x_2)\in B^2:\ (S\circ T)^j x\in E,\ j\ge 0.$ *Then* $\lambda(\Gamma)>0$.

The proof given was a modification of SCHWEIGER [2]. The essential idea is to show that

$$\frac{q_n}{A_n}\ge\gamma>0,\quad \gamma=\gamma(u)\quad\text{a.e.}$$

However in contrast to the Parry-Daniels map it is easy to show that there is a constant $\gamma > 0$ such that for all u

$$\frac{q_n}{A_n} \geq \gamma > 0.$$

From

$$a_{n+1} \leq \frac{q_{n+1}}{q_n}$$

one sees by induction that

$$q_n \leq \left(2 + \frac{1}{q_1} + \cdots + \frac{1}{q_{n-1}}\right) A_n$$

holds. This implies that the set Γ contains a triangle. Therefore the set Γ is less "exceptional" as explained in Remark 2. In fact, Γ contains the triangle with the vertices $(0,0)$, $\left(\frac{1}{2},0\right)$, $\left(\frac{1}{3},\frac{1}{3}\right)$. But it is easy to see that Γ contains at least countably many segments which start at $(0,0)$ but go beyond the line $2x_1 + x_2 = 1$.

The restriction of $S \circ T$ on Γ has the σ-finite invariant measure with density

$$h(x_1, x_2) = \frac{1}{x_1 x_2 (1 - 2x_1 - x_2)}.$$

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