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Effects of rotation on stellar structure and pulsation

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Effects of rotation on stellar structure: rotation induced mixing

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Abstract

Standard models of stellar structure are unable to account for various observational facts, such as the appearance at the surface of chemical elements that have been produced in the nuclear core. Thus there is now a large consensus that some 'extra mixing' must occur in the radiation zones. This mixing is achieved mainly through the shear-turbulence generated by the differential rotation, which itself results from the transport of angular momentum by a large-scale circulation that is induced either by the structural adjustments accompanying the evolution or by the applied torques (stellar wind, accretion, tides). These processes are now being implemented in stellar evolution codes, and they provide a much better agreement with the observations.

The observational evidence

Until recently, stellar models ignored the possibility that some mixing could occur in the radiation zones of stars. But the situation is changing because there is increasing evidence for such mixing, which we shall briefly review.

It is well-known that some A-type stars display anomalies in their surface composition, when they are compared to other, 'normal' stars. These peculiarities were successfully ascribed to radiative acceleration and gravitational settling, by Michaud (1970) and his collaborators. But it turns out that these atomic processes are so efficient that they would produce surface anomalies that are much more pronounced than those observed. For instance, helium would disappear from the surface of A-type stars in about one million years, as was pointed out by Vauclair et al. (1974). Since this is not observed, Vauclair et al. (1978) suggested that some mild turbulence operates near the surface to smooth the composition gradients.

In the Sun also, the profile of the sound velocity below the convection zone, which we know thanks to helioseismology, reveals that the settling of helium is hindered by some mixing.

Another proof of such mixing is the striking flatness of the celebrated Spite plateau, with the ^{7}Li abundance in halo stars depending little on effective temperature (Spite & Spite 1982). If element segregation were alone to operate, the Li depletion would increase with effective temperature, as was recognized by Deliyannis et al. (1990). Therefore one must invoke again some mild mixing in the surface region to enforce the flatness of this plateau.

Finally the overabundance, observed at the surface of massive stars, of chemical elements that are synthesized in the nuclear core (such as He and N), can only be explained if the radiative envelope has undergone some mixing (cf. Meynet & Maeder 2000). Interestingly, these overabundances seem to be correlated with the rotation velocity of the stars (Herrero et al. 1992).

There are thus many indications that radiation zones undergo some kind of mixing, and we believe that its causes have now been identified: these are the large scale circulation required by the transport of angular momentum, and the turbulence generated by the shear of differential rotation.

Meridional circulation

In its original treatment (Eddington 1925; Vogt 1925), the meridional circulation was ascribed to the fact that the radiative flux is no longer divergence-free in a rotating star, due to the centrifugal force. The characteristic time of the circulation was derived by Sweet (1950), and has since been named the Eddington-Sweet time: $t_{\rm ES} = t_{\rm KH}(GM/\Omega^2 R^3)$, with $t_{\rm KH} = GM^2/RL$ being the Kelvin-Helmholtz time. R, M, L designate respectively the radius, mass and luminosity, Ω the angular velocity and G the gravitational constant. Sweet's result suggested that rapidly rotating stars should be well mixed by this circulation, and therefore that they would not evolve to the giant branch, as observed.

However, these early studies overlooked the fact that the circulation carries angular momentum and modifies the rotation profile: starting from initial conditions, the star undergoes a transient phase which lasts indeed about an Eddington-Sweet time, after which it settles into a quasi-stationary regime where the circulation is governed solely by the torques applied to the star. For instance, when the star loses angular momentum through a strong wind, the circulation adjusts precisely such as to transport that momentum to the surface (Zahn 1992). The resulting rotation is then non-uniform, and a baroclinic state sets in, with the temperature varying with latitude along isobars. On the other hand, when the star does not exchange angular momentum, the circulation would die altogether, as predicted by Busse (1982), if it had not to compensate the effects of structural adjustments (contraction, expansion) as the star evolves, and the weak turbulent transport down the gradient of angular velocity that will be discussed next.

Shear turbulence caused by differential rotation

Since the rotation regime that results from the applied torques is not uniform, the shear of that differential rotation is prone to various instabilities, which generate turbulence and therefore mixing. Here we shall consider only those that apparently play a major role, namely the shear instabilities.

Turbulence produced by the vertical shear

Let us first examine the instability produced by the vertical shear, $\Omega(r)$. This instability is very likely to occur, because the Reynolds number characterizing such flows in stars is extremely high, due to the large sizes involved. However the stable entropy stratification acts to hinder the shear instability: in the absence of thermal dissipation, it occurs only if locally

$$\frac{N_T^2}{(\mathrm{d}V_{\rm h}/\mathrm{d}z)^2} \le Ri_{\rm c}, \tag{1}$$

where $V_{\rm h}$ is the horizontal velocity, z the vertical coordinate, and N_T the buoyancy frequency defined by $N_T^2 = (g\delta/H_P)(\nabla_{\rm ad} - \nabla)$, with the classical notations and $\delta = -(\partial \ln \rho/\partial \ln T)_P$. This condition is known as the *Richardson criterion*; the critical Richardson number $Ri_{\rm c}$ is of the order of unity and it depends somewhat on the rotational profile.

In a stellar radiation zone, this criterion is modified because the perturbations are no longer adiabatic, due to thermal diffusion. When the radiative diffusivity K exceeds the turbulent diffusivity $D_{\rm v} = w\ell$ (ℓ and w represent the size and the vertical velocity of the largest eddies), the instability criterion takes the form (Dudis 1974; Zahn 1974)

$$\frac{N^2}{({\rm d}V_{\rm h}/{\rm d}z)^2}\left(\frac{w\ell}{K}\right) \le Ri_{\rm c}. \qquad (2)$$

From the largest eddies that fulfill this condition, one can derive the turbulent diffusivity $D_{\rm v}$ acting in the vertical direction in the radiation zone of a star. However this instability criterion (2) holds only in regions of uniform composition, where the stability is enforced solely by the temperature gradient; when the molecular weight μ increases with depth, it seems at first sight that one should replace this criterion by the original one, expression (1), where now the buoyancy frequency is dominated by the gradient of molecular weight:

$$N^2 \approx N_\mu^2 = \frac{g\varphi}{H_P}\frac{{\rm d}\ln\mu}{{\rm d}\ln P},$$

with $\varphi = (\partial\ln\rho/\partial\ln\mu)_{P,T}$. However, as Meynet and Maeder (1997) pointed out, this condition is so severe that it would prevent any mixing in early-type main sequence stars, contrary to what is observed. We shall see below how that stabilizing action of μ-gradients can be overcome.

Turbulence produced by the horizontal shear

Likewise, the horizontal shear $\Omega(\theta)$ will also generate turbulence, and this turbulence will probably be highly anisotropic, with much stronger transport in the horizontal than in the vertical direction, i.e. $D_{\rm h} \gg D_{\rm v}$. This will thus lead to a 'shellular' rotation state, where the angular velocity depends little on latitude, and where one can assume that $\Omega \sim \Omega(r)$.

Such anisotropic turbulence interferes with the meridional circulation, turning the advective transport into a vertical diffusion (Chaboyer & Zahn 1992). If the vertical velocity of the circulation is given by $u_r(r,\theta) = U(r)P_2(\cos\theta)$, where P_2 is the Legendre polynomial of degree 2, the resulting diffusivity is

$$D_{\rm eff} = \frac{1}{30}\frac{(rU)^2}{D_{\rm h}}, \qquad (3)$$

provided that $D_{\rm h} \ge rU$. Unfortunately, a reliable prescription for that horizontal diffusivity $D_{\rm h}$ is still lacking, in spite of recent attempts to improve it (see Maeder 2003; Mathis et al. 2004).

Another property of such anisotropic turbulence is that, by smoothing out chemical inhomogeneities on level surfaces, it reduces the stabilizing effect of the vertical μ-gradient. The Richardson criterion for the vertical shear instability then involves the horizontal diffusivity D_h: and the vertical component of the turbulent viscosity can be derived as before (Talon & Zahn 1997):

$$D_{\rm v} = Ri_{\rm c}\left[\frac{N_T^2}{K+D_{\rm h}} + \frac{N_\mu^2}{D_{\rm h}}\right]^{-1}\sin^2\theta\left(\frac{{\rm d}\Omega}{{\rm d}\ln r}\right)^2. \qquad (4)$$

Rotational mixing of type I

The two transport processes that have just been discussed (meridional circulation and shear-induced turbulence) are both linked with the differential rotation. Therefore, when modeling the evolution of a star including these mixing processes, it is necessary to calculate also the evolution of its rotation profile $\Omega(r)$ (since Ω is a function of r only, due to the anisotropic turbulence mentioned above). Then all perturbations separate in r and colatitude θ, as illustrated here for the vertical component of the meridional velocity: $u_r(r, \theta) = U(r) P_2(\cos\theta)$. For a detailed account of how this modelization may be implemented, we refer to Zahn (1992), Maeder & Zahn (1998) and Mathis & Zahn (2004).

We first examine the simplest case, that we call 'rotational mixing of type I', where the angular momentum is transported by solely the same processes that are responsible for the mixing, namely meridional circulation and turbulent diffusion. The angular velocity then obeys the following transport equation, obtained by averaging over latitude:

$$\frac{\partial}{\partial t}\left[\rho r^2\,\Omega\right] = \frac{1}{5r^2}\frac{\partial}{\partial r}\left[\rho U r^2\,\Omega\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\rho \nu_{\rm v} r^4 \frac{\partial \Omega}{\partial r}\right] + \text{applied torques}, \tag{5}$$

with $\nu_{\rm v} \approx D_{\rm v}$ given by (4). In spite of the fact that this equation is one-dimensional, it captures the advective character of the angular momentum transport by the meridional circulation: depending on the sense of the circulation, angular momentum may be transported up the gradient of Ω, which is never the case when the effect of meridional circulation is modeled just as a diffusive process, as it is done most often.

The circulation is governed mainly by the applied torques. When the star loses little angular momentum, or none, it settles into a regime of differential rotation where a weak inward flux of angular momentum compensates the turbulent diffusion directed outwards. On the other hand, when the star loses a large amount of angular momentum, the circulation adjusts itself such as to transport precisely that amount towards the surface (Zahn 1992).

Massive main sequence stars belong to the first category, and their models have been seriously improved by the implementation of rotational mixing (Maeder & Meynet 2000). The theoretical isochrones agree with the observed ones, and such rotational mixing accounts well for the observed enhancement of He and N at the surface of early-type stars (Talon et al. 1997; Meynet & Maeder 2000 and subsequent papers). Combined with a suitable description of the mass loss, this type of mixing also predicts the observed proportion of blue and red giants. Finally, such mixing accounts well for the destruction of Li on the blue side of the Li gap, as was shown by Charbonnel and Talon (1999).

The second case applies to solar-type stars, whose modeling has been much less successful, until very recently. In those stars, which lose most of their angular momentum through a magnetized wind (Schatzman 1962), the meridional circulation adjusts so as to carry the required angular momentum towards the surface, at least in the absence of other processes (Zahn 1992). One would then expect that the amount of mixing, and hence the depletion of light elements, be proportional to the loss of angular momentum. This would have consequences that are not confirmed by the observations. To quote the most severe observational test, models that are built according to equation (5), i.e. including only turbulence and meridional circulation, conserve a fast rotating core (Pinsonneault et al. 1989), which is ruled out by helioseismology. Therefore another, more powerful process is responsible for the transport of angular momentum in solar-type stars; by shaping the rotation profile, it will also, though indirectly, determine the extent of mixing.

Rotational mixing of type II

In what we call rotational mixing of type II, the chemical elements are still transported by the meridional circulation and the turbulence caused by differential rotation, but the angular momentum is carried by another process, such as magnetic torquing or internal gravity waves.

Magnetic field

Magnetic fields, because they are almost 'frozen' in the highly conducting stellar material, are very powerful in reducing differential motions. This was already pointed out by Mestel (1953), who claimed that a fossil magnetic field of moderate strength would render stellar rotation nearly uniform. More precisely, in presence of an axisymmetric poloidal field, the angular velocity tends to become uniform along the field lines, a property referred to as Ferraro's law (1937). Thus if the poloidal field lines lie entirely in the radiation zone of a star, they impose uniform rotation there.

The situation changes however when the poloidal field connects with a differentially rotating convection zone, such as that in the Sun. Then this latitude dependent rotation is transmitted along the field lines, and the result is a non uniformly rotating radiation zone, as illustrated by the calculations made by Charbonneau and MacGregor (1993), a behavior that has been confirmed by Brun and Zahn (2006) through 3-dimensional time-dependent calculations. We are thus led to conclude that, in the Sun at least, it is not the magnetic field that is responsible for the uniform rotation of the radiative interior.

In other stars, such as magnetic A-type stars, fossil fields presumably play a much more important role. They could for instance inhibit completely the rotational mixing. However, certain field configurations are unstable, and they could produce MHD turbulence, and possibly mixing, before they relax into a stable state. This is now being explored through numerical simulations.

Internal gravity waves

Since magnetic fields seem unable to enforce uniform rotation in the radiative interior of solar-type stars, we must turn to the other possible mechanism, namely the transport of angular momentum by the internal gravity waves emitted by the turbulent motions at the base of the convection zone.

The restoring force operating on gravity waves is the buoyancy force: therefore they travel only in stably stratified regions, i.e. in radiation zones. There, they transport angular momentum which they deposit wherever they are dissipated through radiative damping. It is by shaping the rotation profile that they indirectly participate in the mixing of chemicals.

This scenario has been tested through numerical simulations performed by Talon et al. (2002), using a rather crude approach with imposed turbulent viscosity. It has since been confirmed through more detailed and more realistic calculations; beside the internal gravity waves, these also include the meridional circulation and the shear induced turbulence (Charbonnel & Talon 2005).

The result is spectacular: internal gravity waves succeed in achieving nearly uniform rotation in solar-type stars at the solar age, as demonstrated by Talon & Charbonnel (2003, 2004, 2005). Furthermore their models predict the observed Li abundances: they explain the Spite plateau for population II stars, and the Li dip in galactic clusters.

Conclusion

To summarize, we are entering a new era of modeling stellar interiors, where rotation will be taken into account, as well as the transport processes operating in the radiation zones. Most pieces of this scheme are now based on robust prescriptions, but some weaknesses remain.

Above all, we need to improve the description of the turbulent transport, in particular that operating in the horizontal direction. Also, the way we handle the generation of internal gravity waves is far from satisfactory. We must clarify whether the gravity waves are able to diffuse chemicals, beside transporting angular momentum. Finally, we have to introduce magnetic fields in our models, at least where we think that they could play a role. In all these subjects, there is much to be expected from the high-resolution numerical simulations that are now undertaken.

But as they stand, the transport processes presented above have already been implemented in several stellar evolution codes. Their treatment will continue to benefit from observational constraints, in particular those we anticipate from asteroseismology, to which this HELAS workshop was dedicated. It is clear that rotational mixing will soon be integrated in the accepted standard model.

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DISCUSSION

Dupret: For which type of stars, stellar evolution and ages is macroscopic mixing due to differential rotation expected to be important?

Zahn: Rotational mixing is expected to play a role nearly everywhere in stellar evolution by changing the internal composition profile.

Grevesse: Solar models including rotational mixing and internal gravity waves fit much better the observations (of internal rotation, and of Be not being destroyed). Do these models suffer the same problem as the standard ones when new solar abundances are used?

Zahn: The theoretical description of rotational mixing has not yet reached that level of sophistication to test its sensitivity to such changes in composition.

Guzik: Since we don't know the spectrum or amplitude or damping of gravity waves, how did Talon & Charbonnel model Li abundance for stars? Were there assumptions made about the gravity waves? Could we infer anything about gravity waves from the observations?

Zahn: Talon & Charbonnel applied the method by Goldreich et al. (1993) to determine the spectrum of gravity waves. That method predicts a spectrum of p-modes that agrees well with the observations. For gravity waves there is also another mechanism, located at the base of convective zone, due to convective overshoot. So it is likely the effect of gravity waves is actually underestimated by Talon & Charbonnel, who take into account only the bulk excitation by Reynolds stresses, as did Goldreich et al.

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Pulsation in rapidly rotating stars

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Abstract

In this review, I discuss pulsational stability of low frequency g-modes and r-modes of rapidly rotating main sequence stars. I also discuss a possible role of low frequency modes in angular momentum transfer in the stars.

Low Frequency Pulsation and Rapid Rotation

Recent detection of low frequency oscillations in rapidly rotating Be stars, by MOST and COROT satellites, has made promising an asteroseismic study of the stars (e.g., Walker et al. 2005, Saio et al. 2007, Cameron et al. 2008). Mode identification and modal stability analysis are two essential parts of the study.

By low frequency oscillations I mean those whose frequency ω in the co-rotating frame of the star is less than $\sqrt{GM/R^3}$, where G is the gravitational constant, and M and R are the mass and radius of the star, respectively. Internal gravity waves (g-modes) and inertial waves such as r-modes are examples of low frequency modes. Since we have for a rapidly rotating star $\Omega \sim \sqrt{GM/R^3}$ with Ω being the angular rotation frequency of the star, the term $m\Omega$ can have a dominant contribution to the frequency $\sigma = \omega - m\Omega$ of a mode observed in an inertial frame so that

$$\sigma \sim -m\Omega, \tag{1}$$

where m denotes the azimuthal wave number of the mode. (Note that since the temporal and azimuthal dependence of oscillation is given by a factor $\exp \mathrm{i}\,(\omega t + m\phi)$ in this paper, if we assume $\omega > 0$, negative and positive values of m correspond to prograde and retrograde modes, respectively.) This suggests that low frequency oscillations of a rapidly rotating star form frequency groups in an inertial frame, to each of which a value of the azimuthal wave number m can be assigned (see, e.g., Walker et al. 2005, Saio et al. 2007, Cameron et al. 2008).

Theoretical analysis of low frequency oscillations in a rapidly rotating star is a difficult task. Rotation cannot be treated as a small perturbation to the oscillation unless $|\Omega/\omega| \ll 1$. Since separation of variables is not possible for oscillations of a rotating star, equations that govern the oscillations become a set of linear partial differential equations, even if we assume the equilibrium structure of the rotating star is axisymmetric about the rotation axis. If we assume that the temporal and azimuthal dependence of the perturbations is given by $\exp \mathrm{i}\,(\omega t + m\phi)$, the oscillation equations are reduced to a set of linear partial differential equations in r and θ. We may count at least three methods to integrate the oscillation equations:

- We can expand the perturbations in terms of spherical harmonic functions $Y_l^m(\theta, \phi)$ for a given m. The radial component of the displacement vector and the Euler perturbation of the pressure, for example, are given by

$$\xi_r = \sum_{j=1}^{j_{\max}} S_{l_j}(r)\, Y_{l_j}^m(\theta,\phi)\, e^{i\omega t}, \quad p' = \sum_{j=1}^{j_{\max}} p'_{l_j}(r)\, Y_{l_j}^m(\theta,\phi)\, e^{i\omega t}, \tag{2}$$

where $l_j = |m| + 2(j-1)$ for even modes and $l_j = |m| + 2j - 1$ for odd modes with $j = 1,\ 2,\ 3,\ \cdots,\ j_{\max}$. Substituting these expansions of finite length into the linearized basic equations, we obtain a set of coupled linear ordinary differential equations of finite dimension for the expansion coefficients such as $S_{l_j}(r)$ and $p'_{l_j}(r)$, which is solved as a boundary-eigenvalue problem by using a standard method (e.g., Lee & Baraffe 1995, Lee 2001; see also Dintrans & Rieutord 2000).

- We may apply the traditional approximation, which makes possible the separation of variables in an approximate way by introducing the separation factor $\lambda_{k,m}(2\Omega/\omega)$, which is an eigenvalue of Laplace's tidal equation, and tends, as $2\Omega/\omega \to 0$, to $l_k(l_k+1)$ with $l_k = |m| + k$ for a positive integer k. Since the perturbations in the traditional approximation are represented by

$$\xi_r = \Xi_k(r)\,\Theta_k^m(\mu;\nu)\, e^{i(m\phi+\omega t)}, \quad p' = \Pi_k(r)\,\Theta_k^m(\mu;\nu)\, e^{i(m\phi+\omega t)}, \tag{3}$$

where $\mu = \cos\theta$, $\nu = 2\Omega/\omega$, and $\Theta_k^m(\mu;\nu)$ denotes an eigenfunction of the Laplace equation, we may classify the modes using the eigenvalue $\lambda_{k,m}$ (e.g., Lee & Saio 1997). The oscillation equations in the traditional approximation are obtained by simply replacing by $\lambda_{k,m}$ the factor $l(l+1)$ in the oscillation equations for a nonrotating star and they may be solved by using a standard numerical method (e.g., Townsend 2005; Dziembowski et al. 2007). See also a review by Gerkema et al. (2008) for the traditional approximation.

- We can directly integrate numerically oscillation equations given as a set of linear partial differential equations in r and θ (e.g., Savonije 2005, 2007).

The frequency spectrum of low frequency modes of a rotating star may be computed in good approximation under the traditional approximation (e.g., Lee & Saio 1987). But, the pulsational stability of the modes cannot be reliably determined by using the traditional approximation, particularly for rapidly rotating stars, because the pulsational stability is influenced by coupling between *traditional* modes associated with $\lambda_{k,m}$ with different indices k for a given m.

In Figures 1 and 2, I show examples of numerical stability analyses of low frequency modes using the traditional approximation (left panel) and the expansion method (right panel) for a $5M_\odot$ main sequence model rotating at a rate $\Omega/\sqrt{GM/R^3} = 0.7$, where the growth rate $\eta = -\omega_{\rm I}/\omega_{\rm R}$ of unstable modes is plotted versus the inertial frame frequency $\bar\sigma$. Note that low frequency modes of slowly pulsating B (SPB) stars are known to be excited by the opacity bump mechanism, and this same mechanism should also be effective to excite low frequency modes in rapidly rotating Be stars (e.g., Lee 2001; Townsend 2005; Sovonije 2005, 2007; Dziembowski et al. 2007). As shown by the figures, a number of both prograde and retrograde g-modes are found unstable under the traditional approximation, but only the prograde g-modes are found unstable if we use the expansion method, which includes the effects of rotational deformation of the equilibrium structure. We also note that odd r-modes are found unstable but even r-modes stable, and that the stability of the r-modes does not strongly depend on the method of calculation. The asymmetry of pulsational stability of prograde and retrograde g-modes in rapidly rotating Be stars has been suggested by Lee (2001), who attributed the cause of the asymmetry to coupling between *traditional* modes associated with $\lambda_{k,m}$ of different indices k for a given m. Savonije (2005, 2007), with his two-dimensional calculation, confirmed that retrograde g-modes of a rapidly rotating star are

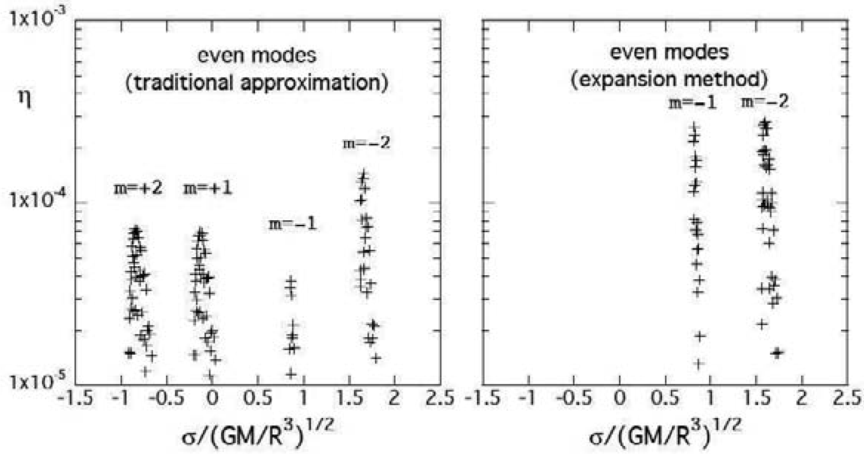

Figure 1: Growth rate $\eta = -\omega_I/\omega_R$ of unstable even g-modes with $|m| = 1$ and 2 for a $5M_\odot$ main sequence model rotating at a rate $\Omega/\sqrt{GM/R^3} = 0.7$. The left and right panels are for modes calculated using the traditional approximation and the expansion method, respectively.

largely stabilized by Coriolis coupling between the g-modes. Comparing stability calculations with and without the effects of rotational deformation of an equilibrium structure, it is found that rotational deformation plays a role in the stabilization of retrograde g-modes although the effects depend on the stellar mass. Since the effects of rotational deformation are included only in an approximate way in the numerical analysis using the expansion method (Lee & Baraffe 1995), an improvement in the treatment of rotational deformation is desirable to make more reliable pulsational stability analyses of rapidly rotating stars.

With MOST experiment, Walker et al. (2005) and Saio et al. (2007) have detected low frequency oscillations in rapidly rotating Be stars. Using the expansion method with the effects of deformation included, they have carried out theoretical calculations of pulsational stability of low m g- and r-modes for rotating B star models. They found that only prograde g-modes and retrograde r-modes, which can be fitted to the detected frequencies except for the very low frequency oscillations, are unstable, and that except for a few g-modes having $\omega \sim \sqrt{GM/R^3}$, almost all retrograde g-modes are stable. Under the traditional approximation, on the other hand, a number of both prograde and retrograde g-modes are found unstable (Dziembowski et al. 2007), and an additional physical effect like visibility of the modes has been considered in order to fit the stability analysis results under the approximation to the detected frequencies (Daszýnska-Daszkiewicz et al. 2007). More theoretical studies will be necessary.

Angular momentum transfer by low frequency oscillations

Using a theory of wave mean-flow interaction as a guide, I discuss the acceleration of a zonal flow in the surface region of a rapidly rotating Be star. Here, I expect that this acceleration takes place as a result of angular momentum transfer (re-distribution) in the interior, which is caused by non-axisymmetric ($m \neq 0$) low frequency modes excited by the opacity bump mechanism, expecting that the accelerated material will be expelled to form a disc around the Be star. In astrophysics, the theory of wave mean-flow interaction has been applied to various problems, including synchronization between the orbital motion and stellar rotation in

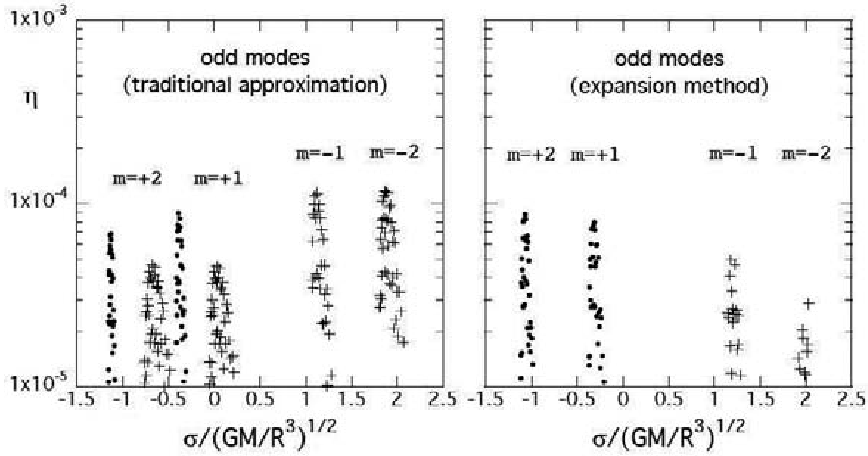

Figure 2: Growth rate $\eta = -\omega_{\rm I}/\omega_{\rm R}$ for unstable odd g- and r-modes with $|m| = 1$ and 2 for a $5M_\odot$ main sequence model rotating at a rate $\bar{\Omega} = 0.7$, where the pluses and the filled circles are for g-modes and r-modes, respectively. The left and right panels are for the results obtained by applying the traditional approximation and the expansion method, respectively.

a binary system (Zahn 1975, 1977), Be phenomena (Ando 1982, 1983, 1986), and evolution of the interior rotation velocity of the sun (e.g., Charbonnel & Talon 2005).

Let us decompose, e.g., the velocity field of a fluid element in a star as

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)}, \tag{4}$$

where $\mathbf{v}^{(0)}$ is time-independent, and $\mathbf{v}^{(1)}$ is the wave component assumed to satisfy the condition:

$$\overline{\mathbf{v}^{(1)}} \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathbf{v}^{(1)} d\phi = 0, \tag{5}$$

where the bar indicates the zonal average. The mean-flow component is given by

$$\overline{\mathbf{v}} = \overline{\mathbf{v}^{(0)} + \mathbf{v}^{(2)}}, \tag{6}$$

where $\mathbf{v}^{(2)}$ is a slowly changing part in a time scale much longer than the periods of the waves. If we let a denote a measure of wave amplitudes, we have $\mathbf{v}^{(1)} \sim O(a)$ and $\mathbf{v}^{(2)} \sim O(a^2)$ as $a \to 0$. Substituting such decomposition of physical quantities into the basic equations, we obtain a set of partial differential equations for the mean-flow quantities. As a result of zonal averaging, the resultant equations contain the second-order terms like $\overline{v_i^{(1)} v_j^{(1)}}$, where the wave quantities such as $\mathbf{v}^{(1)}$ are assumed known as a solution to linear wave equations. An example of a set of mean-flow equations may be found in Andrews & McIntyre (1976). It is difficult to solve the mean-flow equations in general.

For a rotating star, it will be reasonable to assume that

$$\mathbf{v}^{(0)} = \left(0,\ 0,\ ,v_\phi^{(0)} = r\sin\theta\Omega\right), \quad \mathbf{v}^{(1)} = \mathbf{v}' = i\omega\boldsymbol{\xi} - r\sin\theta\,(\boldsymbol{\xi}\cdot\nabla\Omega)\,\mathbf{e}_\phi, \tag{7}$$

where $\boldsymbol{\xi}$ is the displacement vector.

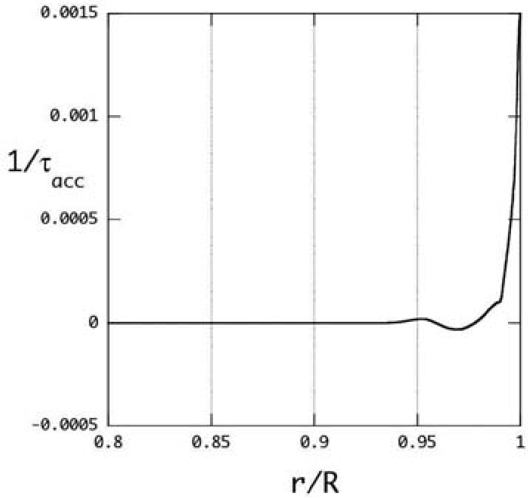

Figure 3: $1/\tau_{\rm acc}$ versus r/R for a $m=1$ r-mode of a $5M_\odot$ main sequence star rotating at $\Omega/\sqrt{GM/R^3}=0.7$. The amplitude normalization is given by $S_{l_1}(R)/R=1$.

The evolution equation of angular momentum in an inertial frame may be given by (e.g., Andrews & McIntyre 1976)

$$\frac{\partial}{\partial t}\left(r\sin\theta\bar\rho\bar v_\phi\right)=-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(r\sin\theta\bar\rho S^{\rm EP}_{r\phi}\right)\right]-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(r\sin\theta\bar\rho S^{\rm EP}_{\theta\phi}\right)\right], \quad (8)$$

where the Eliassen-Palm flux $(S^{\rm EP}_{r\phi}, S^{\rm EP}_{\theta\phi})$ is defined by

$$S^{\rm EP}_{r\phi}=\overline{v'_r v'_\phi}+\overline{2\Omega\cos\theta\frac{v'_\theta s'}{ds/dr}}, \quad S^{\rm EP}_{\theta\phi}=\overline{v'_\theta v'_\phi}-\overline{2\Omega\sin\theta\frac{v'_\theta s'}{ds/dr}}, \quad (9)$$

and s denotes the specific entropy. For non-dissipative and non-transient waves, it has been argued that the right-hand side of equation (8) vanishes and hence no acceleration of the zonal flow occurs, which is known as a non-acceleration theorem (e.g., Dunkerton 1980). A non-acceleration theorem for the astrophysical application has been proved by Goldreich & Nicolson (1989). Assuming $\xi_r=-s'/(ds/dr)$, and integrating over the variable θ, we obtain

$$\frac{\partial}{\partial t}\left\langle r\sin\theta\bar\rho\bar v_\phi\right\rangle=-\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left\langle r\sin\theta\bar\rho\, S_{r\phi}\right\rangle\right], \quad (10)$$

where

$$S_{r\phi}=\overline{v'_r\left(v'_\phi+2\Omega\cos\theta\xi_\theta\right)}, \quad (11)$$

and

$$\langle f\rangle=\frac{1}{2}\int_0^{2\pi} f\sin\theta d\theta. \quad (12)$$

The expression (11) for the flux is the same as that proposed by Pantillon et al. (2007). Equation (10) is our basic equation to discuss angular momentum transfer (redistribution) in the stellar interior by non-axisymmetric ($m\neq 0$) modes of a rotating star.

It is useful to note that in the limit of $\Omega \rightarrow 0$, we have

$$\frac{\partial}{\partial t}\left\langle r\sin\theta\bar{\rho}\bar{v}_\phi\right\rangle = \frac{m}{2}\frac{1}{4\pi r^2}\frac{dw}{dr} \qquad (13)$$

where w denotes the work integral. This equation suggests that the acceleration of a zonal flow (rotation) takes place in a driving region ($dw/dr > 0$) for a retrograde ($m > 0$) mode but in a damping region ($dw/dr < 0$) for a prograde ($m < 0$) mode. For oscillations of rapidly rotating stars, we do not necessarily obtain a relation similar to equation (13), which can still be a good approximation.

Low frequency oscillations observed in rapidly rotating Be stars are excited by the opacity bump mechanism, and they suffer strong nonadiabatic effects like excitation and damping of waves in the surface region of the stars. It is convenient to define the acceleration time scale $\tau_{\rm acc}$ by

$$\frac{1}{\tau_{\rm acc}} = \frac{\partial}{\partial t}\ln\left\langle r\sin\theta\bar{\rho}\bar{v}_\phi\right\rangle, \qquad (14)$$

which measures the strength of acceleration (deceleration) of the zonal flow. As an example, in Figure 3, we plot $1/\tau_{\rm acc}$ versus the fractional radius r/R for a $m = 1$ r-mode of a $5M_\odot$ main sequence model rotating at $\Omega/\sqrt{GM/R^3} = 0.7$, where the amplitude normalization is given by $S_{l_1}(R)/R = 1$. A strong acceleration occurs at the surface. We can obtain similar behavior of $1/\tau_{\rm acc}$ for unstable prograde g-modes. To determine the efficiency of the acceleration caused by many unstable g- and r-modes, we need to know amplitudes of the excited modes, which requires a nonlinear theory of oscillations of a rotating star (e.g., Schenk et al. 2002).

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Wave transport in differentially rotating stellar radiation zones

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Abstract

In this work, the complete interaction between low-frequency internal gravity waves and differential rotation in stably strongly stratified stellar radiation zones is examined. First, the modification of the structure of those waves due to the Coriolis acceleration is obtained. Then, their feed-back on the angular velocity profile through their induced angular momentum transport is derived. Finally, perspectives are discussed.

Motivation

Internal Gravity Waves (hereafter IGWs) are now considered as an essential transport mechanism in (differentially) rotating stellar radiation zones which are the seat of the mixing during star evolution (cf. Talon & Charbonnel 2005). Furthermore, they could be excited by turbulent movements induced by adjacent convective regions at low frequencies ($\sim 1\mu$Hz in the Sun) that are of the order of the inertial one (2Ω, Ω being the star's angular velocity). The Coriolis acceleration is thus an essential restoring force for the wave dynamics as the buoyancy one associated with the stable stratification. Moreover, IGWs are excited and propagate in regions that are differentially rotating both in the radial and in the latitudinal directions. This is the reason why we undertake in this work the treatment of the complete interaction between the low-frequency IGWs and the differential rotation, which is chosen to be the more general as possible ($\Omega\,(r,\theta)$). We derive their spatial structure modified by the Coriolis acceleration and their feedback on the angular velocity profile through their induced angular momentum transport.

Low-frequency IGWs in differentially rotating radiation zones

To treat the IGWs dynamics in a differentially rotating star, we have to solve the complete inviscid system formed by the momentum equation

$$\left(\partial_t+\Omega\partial_\varphi\right)\vec{u}+2\Omega\,\widehat{e}_z\times\vec{u}+r\sin\theta\left(\vec{u}\cdot\vec{\nabla}\Omega\right)\widehat{e}_\varphi=-\frac{1}{\overline{\rho}}\vec{\nabla}\widetilde{P}-\vec{\nabla}\widetilde{\Phi}+\frac{\widetilde{\rho}}{\overline{\rho}^2}\vec{\nabla}\overline{P}, \tag{1}$$

the continuity equation $\left(\partial_t+\Omega\partial_\varphi\right)\widetilde{\rho}+\vec{\nabla}\cdot\left(\overline{\rho}\vec{u}\right)=0$, the energy transport equation which we give here in the adiabatic limit

$$\left(\partial_t+\Omega\partial_\varphi\right)\left(\frac{\widetilde{P}}{\Gamma_1\overline{P}}-\frac{\widetilde{\rho}}{\overline{\rho}}\right)+\frac{N^2}{\overline{g}}u_r=0 \tag{2}$$

and the Poisson's equation $\nabla^2\widetilde{\Phi} = 4\pi G\widetilde{\rho}$. ρ, Φ, P are respectively the fluid density, gravific potential and pressure. Each of them has been expanded as: $X(r,\theta,\varphi,t) = \overline{X}(r) + \widetilde{X}(r,\theta,\varphi,t)$ where $\overline{X}$ is the mean hydrostatic value of X on the isobar, $\widetilde{X}$ being its wave's associated fluctuation. $N^2 = \overline{g}\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln\overline{P}}{\mathrm{d}r} - \frac{\mathrm{d}\ln\overline{\rho}}{\mathrm{d}r}\right)$ is the Brunt-Vaïsälä frequency where $\Gamma_1 = (\partial\ln P/\partial\ln\rho)_S$ (S being the macroscopic entropy) is the adiabatic exponent. $\vec{u}$ is the wave velocity field. Finally, (r,θ,φ) are the usual spherical coordinates with their unit vector basis $\{\widehat{e}_k\}_{k=r,\theta,\varphi}$ while $\widehat{e}_z = \cos\theta\,\widehat{e}_r - \sin\theta\,\widehat{e}_\theta$ is the one along the rotation axis. t is the time and G the universal gravity constant. To solve this system, three main approximations can be assumed:

- *the JWKB approximation*: waves which are studied here are low-frequency ones such that $\sigma \ll N$ (σ is the wave frequency in an inertial reference frame; see Talon & Charbonnel 2005 and Pantillon et al. 2007 for a detailed discussion of their spectrum). Then, the JWKB approximation can be adopted.

- *the Traditional approximation*: stellar radiation zones are stably strongly stratified regions. Then, in the case where the angular velocity (Ω) is reasonably weak compared to the break-down one, $\Omega_K = \sqrt{G\,M/R^3}$ (M and R being respectively the star's mass and radius), we are in a situation where the centrifugal acceleration can be neglected to the first order and where $2\Omega \ll N$. This allows to adopt the Traditional approximation where the latitudinal component (along $\widehat{e}_\theta$) of the rotation vector $\vec{\Omega} = \Omega\,\widehat{e}_z = \Omega_V\widehat{e}_r + \Omega_H\widehat{e}_\theta$ (with $\Omega_V = \Omega\cos\theta$ and $\Omega_H = -\Omega\sin\theta$) can be neglected for all latitudes.

Let us present a brief local analysis of this approximation in the simplest case of a uniform rotation (see also Lee & Saio 1997). The wave vector $\vec{k}$ and Lagrangian displacement $\vec{\xi}$ are expanded as

$$\vec{k} = k_V\widehat{e}_r + \vec{k}_H \quad \text{and} \quad \vec{\xi} = \xi_V\widehat{e}_r + \vec{\xi}_H, \tag{3}$$

where $\vec{k}_H = k_\theta\widehat{e}_\theta + k_\varphi\widehat{e}_\varphi$, $k_H = |\vec{k}_H|$, $\vec{\xi}_H = \xi_\theta\widehat{e}_\theta + \xi_\varphi\widehat{e}_\varphi$, $\xi_H = |\vec{\xi}_H|$ and $\vec{\xi} \propto \exp\left[i\left(\vec{k}\cdot\vec{r} - \sigma t\right)\right]$.

For low-frequency waves in radiation zones, we can write $\vec{k}\cdot\vec{\xi} = k_V\xi_V + \vec{k}_H\cdot\vec{\xi}_H \approx 0$ since $\vec{\nabla}\cdot\left(\overline{\rho}\,\vec{\xi}\right) \approx 0$ (this is the anelastic approximation that filters out acoustic waves which have higher frequencies), from which we deduce that $\xi_V/\xi_H \approx -k_H/k_V$.

Next, using the results given in Unno et al. (1989), the dispersion relation for the low-frequency gravito-inertial waves is obtained:

$$\sigma^2 \approx N^2\frac{k_H^2}{k^2} + \frac{\left(2\vec{\Omega}\cdot\vec{k}\right)^2}{k^2}, \tag{4}$$

where the two terms correspond respectively to the dispersion relations of IGWs and of inertial waves. In the case where $2\Omega \ll N$ and $\sigma \ll N$ the previous dispersion relation gives $k_H^2/k^2 \ll 1$. The vertical wave vector is then larger than the horizontal one while the displacement vector is almost horizontal: $|k_H| \ll |k_V|$, $|\xi_V| \ll |\xi_H|$. On the other hand, we get $\left(2\vec{\Omega}\cdot\vec{k}\right)^2 \approx (2\Omega_V k_V)^2$. The latitudinal component of the rotation vector can thus be neglected in whole the sphere.

A global demonstration in spherical geometry is given in Friedlander (1987) who gives the frequency domain of application of this approximation in the case of uniform rotation ($2\Omega < \sigma \ll N$) which is also discussed in Mathis et al. (2008). Its validy domain in the case of a general differential rotation law will be discussed hereafter.

- *the quasi-adiabatic approximation*: Following Press (1981) and Zahn et al. (1997), we adopt the quasi-adiabatic approximation to treat the thermal damping of IGWs. Let us recall here that this damping is responsible for the net transport of angular momentum which is due to bias in the wave's Doppler shift by differential rotation between retrograde ($m > 0$)

and prograde waves ($m<0$)[1] that transport respectively a negative and a positive flux of angular momentum (see Eq. 16 and Goldreich & Nicholson 1989).

Under those approximations, and assuming the anelastic one ($\vec{\nabla}\cdot(\overline{\rho}\vec{u})=0$), the wave's velocity field is then obtained (the details of the derivation are given in Mathis 2008):

$$\vec{u}\left(\vec{r},t\right)=\sum_{k=\{r,\theta,\varphi\}}\left[\sum_{\sigma,m,j}u_{k;j,m}\left(\vec{r},t\right)\right]\widehat{e}_k \tag{5}$$

where

$$u_{r;j,m}(\vec{r},t)=\frac{\widehat{\sigma}}{N}\frac{\lambda_{j,m}^{1/2}\left(r;\widehat{\nu}\right)}{r}w_{j,m}\left(r,\theta;\widehat{\nu}\right)\sin\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right), \tag{6}$$

$$u_{\theta;j,m}(\vec{r},t) = -\frac{\widehat{\sigma}}{r}\mathcal{G}^{\theta}_{j,m}\left(r,\theta;\widehat{\nu}\right)\cos\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right), \tag{7}$$

$$u_{\varphi;j,m}(\vec{r},t) = \frac{\widehat{\sigma}}{r}\mathcal{G}^{\varphi}_{j,m}\left(r,\theta;\widehat{\nu}\right)\sin\left[\Psi_{j,m}\left(r,\varphi,t\right)\right]D_{j,m}\left(r,\theta;\widehat{\nu}\right). \tag{8}$$

The "local" frequency ($\widehat{\sigma}$) [2] which accounts for the Doppler shift by the differential rotation and the "spin parameter" (see Lee & Saio 1997) are defined:

$$\widehat{\sigma}\left(r,\theta\right)=\sigma+m\Omega\left(r,\theta\right)\quad\text{and}\quad\widehat{\nu}\left(r,\theta\right)=\frac{2\Omega\left(r,\theta\right)}{\widehat{\sigma}\left(r,\theta\right)}=R_o^{-1}, \tag{9}$$

where R_o is the Rossby number. Unlike the case of uniform rotation, variables do not separate neatly anymore in the case of general differential rotations $\Omega\left(r\right)$ and $\Omega\left(r,\theta\right)$. The velocity components are thus expressed in terms of the 2D dynamical pressure ($P/\overline{\rho}$) eigenfunctions $w_{j,m}$ which are solutions of the following eigenvalue equation:

$$\mathcal{O}_{\widehat{\nu};m}\left[w_{j,m}\left(r,x;\widehat{\nu}\right)\right]=-\lambda_{j,m}\left(r;\widehat{\nu}\right)w_{j,m}\left(r,x;\widehat{\nu}\right) \tag{10}$$

where we define the General Laplace Operator (GLO)

$$\begin{aligned}\mathcal{O}_{\widehat{\nu};m} &= \frac{1}{\widehat{\sigma}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\left(1-x^2\right)}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)}\frac{\mathrm{d}}{\mathrm{d}x}\right]-\frac{m}{\widehat{\sigma}^2\mathcal{D}(r,x;\widehat{\nu})}\left(1-x^2\right)\frac{\partial_x\Omega}{\widehat{\sigma}}\frac{\mathrm{d}}{\mathrm{d}x}\\ &\quad-\frac{1}{\widehat{\sigma}}\left[\frac{m^2}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)\left(1-x^2\right)}+m\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\widehat{\nu}x}{\widehat{\sigma}\mathcal{D}\left(r,x;\widehat{\nu}\right)}\right)\right]\end{aligned} \tag{11}$$

with

$$\mathcal{D}\left(r,x;\widehat{\nu}\right)=1-\widehat{\nu}^2x^2+\widehat{\nu}\left(\partial_x\Omega/\widehat{\sigma}\right)x\left(1-x^2\right) \tag{12}$$

and $x=\cos\theta$. $\mathcal{O}_{\widehat{\nu};m}$ is the generalization of the classical Laplace tidal operator (Laplace 1799), the eigenfunctions $w_{j,m}$ being thus a generalization of the Hough functions (Hough 1898, Ogilvie & Lin 2004). $\lambda_{j,m}\left(r;\widehat{\nu}\right)$ are its eigenvalues; here, we focus on positive ones that correspond to propagative waves (cf. Ogilvie & Lin 2004). The GLO is a differential operator in x only and the $w_{j,m}$ form a complete orthogonal basis

$$\int_{-1}^{1}w^*_{i,m}\left(r,x;\widehat{\nu}\right)w_{j,m}\left(r,x;\widehat{\nu}\right)\mathrm{d}x=\mathcal{C}_{i,m}\delta_{i,j}, \tag{13}$$

[1] The wave phase is expanded as $\exp\left[i\left(m\varphi+\sigma t\right)\right]$.

[2] Note that $\widehat{\sigma}$ can vanish that corresponds to corotation resonance. In layer(s) where this happens (which are called critical layers), a careful treatment of the complete fluid dynamics equations has to be undertaken that is out of the scope of the present paper (see Booker & Bretherton 1967).

where $\mathcal{C}_{i,m}$ is the normalization factor and $\delta_{i,j}$ is the usual Kronecker symbol. The dispersion relation is then given by

$$k^2_{V;j,m}(r) = \frac{\lambda_{j,m}(r;\widehat{\nu})\,N^2}{r^2} \tag{14}$$

where $k_{V;j,m}$ is the vertical component of the wave vector ($\lambda_{j,m}$ has the dimension of $\left[t^2\right]$). That leads to the following expressions for the JWKB phase function

$$\Psi_{j,m}(r,\varphi,t) = \sigma t + \int_r^{r_c} k_{V;j,m}\,\mathrm{d}r' + m\varphi \tag{15}$$

(r_c is the radius of the basis (or the top) of the adjacent convective region that excites the waves) and for the damping term

$$D_{j,m} = \exp\left[-\frac{\tau_{j,m}(r,\theta;\widehat{\nu})}{2}\right] \text{ where } \tau_{j,m} = \int_r^{r_c} K\frac{\lambda^{3/2}_{j,m}(r;\widehat{\nu})\,N^3}{\widehat{\sigma}}\frac{\mathrm{d}r'}{r'^3}, \tag{16}$$

K being the thermal diffusivity. On the other hand, the latitudinal and azimuthal eigenfunctions are defined

$$\mathcal{G}^{\theta}_{j,m}(r,x;\widehat{\nu}) = \frac{1}{\widehat{\sigma}^2}\frac{1}{\mathcal{D}(r,x;\widehat{\nu})\sqrt{1-x^2}}\left[-\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + m\widehat{\nu}x\right]w_{j,m} \tag{17}$$

$$\begin{aligned}\mathcal{G}^{\varphi}_{j,m}(r,x;\widehat{\nu}) &= \frac{1}{\widehat{\sigma}^2}\frac{1}{\mathcal{D}(r,x;\widehat{\nu})\sqrt{1-x^2}}\\ &\times \left[-\left(\widehat{\nu}x - \left(1-x^2\right)\frac{\partial_x\Omega}{\widehat{\sigma}}\right)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + m\right]w_{j,m}.\end{aligned} \tag{18}$$

As it has been emphasized by Mathis et al. (2008) and references therein, the Traditional approximation has to be used carefully since it modifies the mathematical properties of the adiabatic wave operator. Here, in the case of a general differential rotation law, it is applicable in spherical shell(s) such that $\mathcal{D} > 0$ everywhere ($\forall\, r$ and $\forall\, \theta \in [0,\pi]$). There, the adiabatic wave operator is elliptic corresponding to regular (elliptic) gravito-inertial waves (see Dintrans & Rieutord 2000 for a detailed classification of such waves). In the other spherical shell(s), where both $\mathcal{D} < 0$ and $\mathcal{D} > 0$, the adiabatic wave operator is hyperbolic and the Traditional approximation cannot be applied because of the adiabatic wave's velocity field (and wave operator) singularity where $\mathcal{D} = 0$. Regularization is allowed there by thermal and viscous diffusions that lead to shear layers, the attractors, where strong dissipation occurs that can induce transport and mixing. In Fig. 2, we illustrate for a given chosen theoretical angular velocity profile (cf. Fig. 1) how those two types of spherical shells (respectively where the Traditional approximation is allowed or forbidden) could appear.

Transport of angular momentum

Since the complete wave's velocity field is derived, we focus on the induced transport of angular momentum. The vertical and horizontal Lagrangian angular momentum fluxes are respectively defined:

$$\begin{aligned}\mathcal{F}^{\mathrm{AM}}_V(r,\theta) &= \overline{\rho} r\sin\theta\int_\sigma \left\langle u_r u_\varphi + 2\Omega\cos\theta u_r\xi_\theta\right\rangle_\varphi \mathrm{d}\sigma\\ \text{and}\quad \mathcal{F}^{\mathrm{AM}}_H(r,\theta) &= \overline{\rho} r\sin\theta\int_\sigma \left\langle u_\theta u_\varphi\right\rangle_\varphi \mathrm{d}\sigma\end{aligned} \tag{19}$$

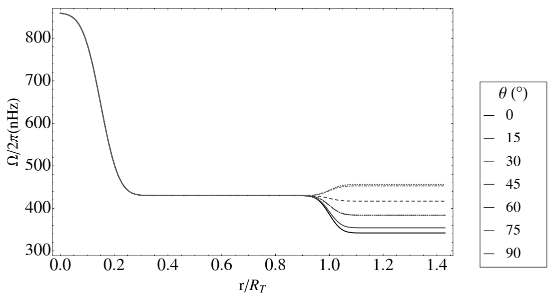

Figure 1: Synthetic internal rotation profile as it may be in the Sun (cf. García et al. 2007): $\Omega_{\rm syn}(r,\theta) = \Omega_s + \Omega_s A_c \left[1 - {\rm Erf}\left((r-R_c)/l_c\right)\right] + 1/2\left[1 + {\rm Erf}\left((r-R_T)/l_T\right)\right]\left(A + B\cos^2\theta + C\cos^4\theta - \Omega_s\right)$, where $\Omega_s = 430\,{\rm nHz}$, $A_c = 1/2$ (such that $\Omega_{\rm syn}(0,\theta) = 2\Omega_s$), $R_c = 0.15 R_T$, $l_c = 0.075 R_T$, $R_T = 0.71 R_\odot$ (the position of the Tachocline), $l_T = 0.05 R_T$, $A = 456\,{\rm nHz}$, $B = -42\,{\rm nHz}$ and $C = -72\,{\rm nHz}$ (we assume here a Tachocline that is thicker than in reality).

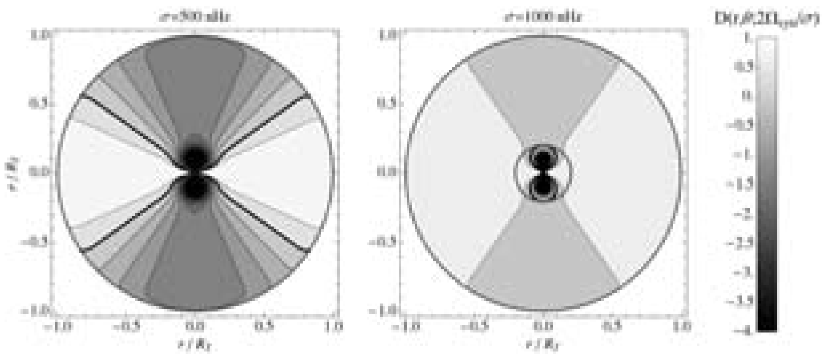

Figure 2: $\mathcal{D}(r,\theta;2\Omega_{\rm syn}/\sigma)$ as a function of r and θ (cf. Eq. 12) for $\sigma = 500$ nHz (Left) and $\sigma = 1000$ nHz (Right) for axisymmetric waves ($m = 0$). The critical surface $\mathcal{D}(r,\theta;2\Omega_{\rm syn}/\sigma) = 0$ is given by the thick black line and the iso-$\mathcal{D}$ lines such that $\mathcal{D}(r,\theta;2\Omega_{\rm syn}/\sigma) > 0$ and $\mathcal{D}(r,\theta;2\Omega_{\rm syn}/\sigma) < 0$ are respectively given by the red and the blue lines. The Traditional Approximation (T. A.) applies in spherical shell(s) such that $\mathcal{D} > 0$ everywhere ($\forall\, r$ and $\forall\,\theta \in [0,\pi]$); there, waves are regular at all latitudes. In other spherical shell(s), where both $\mathcal{D} > 0$ and $\mathcal{D} < 0$, the T. A. does not apply due to the singularity at $\mathcal{D} = 0$. Therefore, for $\Omega_{\rm syn}$, the T. A. does not apply for $\sigma = 500$ nHz while it applies for $\sigma = 1000$ nHz in the external spherical shell with the inner border given by the thick red circle.

where $\langle\cdots\rangle_\varphi = (1/2\pi)\int_0^{2\pi}\cdots{\rm d}\varphi$, where the Lagrangian wave displacement is defined such that: $\vec{u} = (\partial_t + \Omega\partial_\varphi)\vec{\xi} - r\sin\theta\left(\vec{\xi}\cdot\vec{\nabla}\Omega\right)\widehat{e}_\varphi$ and where we sum over the excited spectrum. Using Eqs. (7-8), we get $\mathcal{F}_H^{\rm AM} = 0$. Then, following the methodology given in Zahn et al. (1997), Pantillon et al. (2007) and Mathis et al. (2008), we get the vertical action of angular momentum which is conserved in the adiabatic limit

$$\mathcal{L}_V^{\mathrm{AM}}\left(r,x;\widehat{\nu}\right)=r^2\mathcal{F}_V^{\mathrm{AM}}$$
$$=-r_c^2\int_{\sigma}\sum_{m,j}\left\{\frac{\widehat{m}_{j,m}\left(r_c,x;\widehat{\nu}_c\right)}{\widehat{\sigma}_{\mathrm{CZ}}}\mathcal{F}_{V;j,m}^{\mathrm{E}}\left(r_c,x;\widehat{\nu}_c\right)D_{j,m}^2\right\}\mathrm{d}\sigma. \tag{20}$$

r_c is the radius of the basis (or the top) of the adjacent convective region that excites the waves while $\widehat{\nu}_c=2\Omega_{\mathrm{CZ}}\left(r_c,\theta\right)/\widehat{\sigma}_{\mathrm{CZ}}$ where $\widehat{\sigma}_{\mathrm{CZ}}=\sigma+m\Omega_{\mathrm{CZ}}\left(r_c,\theta\right)$, Ω_{CZ} being its angular velocity. On the other hand, $\mathcal{F}_{V;j,m}^{\mathrm{E}}\left(r_c,x;\widehat{\nu}_c\right)$ is the monochromatic energy flux injected by turbulent convective movements at $r=r_c$ in the studied radiation zone and

$$\widehat{m}_{j,m}\left(r,x;\widehat{\nu}\right)=\frac{\sin\theta\,\widehat{\sigma}^2\,w_{j,m}\left[\mathcal{G}_{j,m}^{\varphi}-\widehat{\nu}\cos\theta\,\mathcal{G}_{j,m}^{\theta}\right]}{w_{j,m}^2} \tag{21}$$

is the 2D function which describes its conversion into angular momentum flux.

Following Mathis & Zahn (2005), averaging over latitudes Ω and $\mathcal{L}_V^{\mathrm{AM}}$ in spherical shell(s) where the Traditional approximation applies and expanding this former as $\mathcal{L}_V^{\mathrm{AM}}=\sum_l\mathcal{L}_{V;l}^{\mathrm{AM}}\left(r\right)\sin^2\theta P_l\left(\cos\theta\right)$, we get for the mean rotation rate on an isobar $\left(\left\langle\Omega\right\rangle_\theta\right)$

$$\overline{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\left\langle\Omega\right\rangle_\theta\right)-\frac{1}{5r^2}\partial_r\left(\overline{\rho}r^4\left\langle\Omega\right\rangle_\theta U_2\right)=\frac{1}{r^2}\partial_r\left(\overline{\rho}\nu_V r^4\partial_r\left\langle\Omega\right\rangle_\theta\right)-\frac{1}{r^2}\partial_r\left[\left\langle\mathcal{L}_V^{\mathrm{AM}}\right\rangle_\theta\right], \tag{22}$$

and for the first mode of the latitudinal rotation

$$\overline{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\Omega_2\right)-2\overline{\rho}\left\langle\Omega\right\rangle_\theta\left[2V_2-\frac{1}{2}\frac{\mathrm{d}\ln\left(r^2\left\langle\Omega\right\rangle_\theta\right)}{\mathrm{d}\ln r}U_2\right]$$
$$=\frac{1}{r^2}\partial_r\left(\overline{\rho}\nu_V r^4\partial_r\Omega_2\right)-10\,\overline{\rho}\nu_H\Omega_2-\frac{1}{r^2}\partial_r\left[\mathcal{L}_{V;2}^{\mathrm{AM}}\left(r\right)\right], \tag{23}$$

where $\widetilde{\Omega}_2\left(r,\theta\right)=\Omega_2\left(r\right)\left[P_2\left(\cos\theta\right)+1/5\right]$ and $\Omega=\left\langle\Omega\right\rangle_\theta+\widetilde{\Omega}_2$.

The meridional circulation is expanded in Legendre polynomials as $\vec{\mathcal{U}}_{\mathrm{M}}\left(r,\theta\right)=\sum_{l>0}\left\{U_l\left(r\right)P_l\left(\cos\theta\right)\widehat{e}_r+V_l\left(r\right)\partial_\theta P_l\left(\cos\theta\right)\widehat{e}_\theta\right\}$ while $\left(\nu_V,\nu_H\right)$ are respectively the vertical and the horizontal turbulent viscosities and $\mathrm{d}/\mathrm{d}t$ is the Lagrangian derivative that accounts for the contractions and the dilatations of the star during its evolution.

Those equations give the evolution of the differential rotation, both in the radial and in the latitudinal directions, in the spherical shell(s) where the Traditional approximation can be applied. This is the first time that an evolution equation for differential rotation (both in r and θ) capturing gravito-inertial waves feedback is derived, taking into account the modification of IGWs through the Coriolis acceleration and their feedback on the angular velocity profile through the net induced transport of angular momentum due to the differential damping of retrograde and prograde waves.

Conclusion

In this work, a complete formalism to treat the dynamics of regular (elliptic) low-frequency gravito-inertial waves in stably strongly stratified differentially rotating stellar radiation zones, from Tachocline(s) where they are excited to their bulk, has been derived. Their feedback on the angular velocity profile through the induced angular momentum transport is then treated. Future works must be devoted to its implementation in existing dynamical stellar evolution codes and to its application to different type of stars and evolution stages. This effort will lead to the building of more and more realistic stellar models which will benefit from new constraints provided by the development of asteroseismology both on the ground and in space.

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(in back) Konstanze Zwintz, Paul Beck and Wojciech Dziembowski

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The asymptotic structure of the p-modes frequency spectrum in rapidly rotating stars

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Abstract

An asymptotic theory for p-modes in rapidly rotating stars is constructed by studying the acoustic ray dynamics. The dynamics is then interpreted in terms of mode properties, using the concepts and methods developed in the field of quantum chaos. Accordingly, the high-frequency spectrum is a superposition of regular frequency patterns associated with the near-integrable regions of the ray dynamics phase space and an irregular frequency subset associated with a chaotic phase space region. Estimation of the mode visibilities from the disk-averaging factor suggests that the visible spectrum mainly contains a class of regular modes and the irregular frequency subset.

Introduction

Though approximate, the asymptotic theory of p-modes in slowly rotating stars plays an important rôle in the interpretation of solar and stellar pulsation data. In particular, the frequency regular patterns described by the theory constitute crucial a priori information for mode identification. The theory initially built for adiabatic pulsations in nonrotating stars has then been extended to rotating stars under the assumption that the rotation effects are small enough to be treated as a perturbation. The range of validity of this assumption was not known, however. Accurate numerical calculations of p-modes fully taken into account the centrifugal deformation of polytropic uniformly rotating stars have shown that the asymptotic structure of the nonrotating acoustic frequency spectrum is destroyed above a certain rotation rate (Lignières et al. 2006). For example, the so-called small frequency separation is no longer small at these rotation rates. This breakdown of the usual spectrum organization may explain, at least partly, the difficulty to interpret the frequencies observed in rapidly rotating stars. The same calculations also revealed that at high rotation rates some modes, those that were low degree and high order (say $n \geq 5$) modes at zero rotation, display a new form of organization with new types of regular frequency spacings. This result has since been confirmed in calculations including the Coriolis force and the gravitational potential perturbation, and it has been extended to non-axisymmetric modes (Reese et al. 2008). Moreover, above $\Omega = 0.11\Omega_K$, these regular patterns provide a better mode identification than perturbative methods (Ω_K is the Keplerian rotation rate). These results have been obtained for polytropic model of stars but a recent work by Reese et al. (2009) seems to confirm them for the more realistic models of rotating stars of MacGregor et al. (2007). It remains that this regular behaviour had been found empirically by analyzing numerically computed p-modes and needed to be better understood, possibly in the framework of an asymptotic theory.

The asymptotic theory

To build an asymptotic theory for p-modes in rotating stars, the starting point is the same as for nonrotating stars: it consists in applying the WKB approximation to the equation governing the adiabatic perturbation. Accordingly, wave-like solutions $A(x)\exp(i\Psi(x) - i\omega t)$ are sought under the assumption that their wavelength is much smaller than the typical lengthscale of the background medium. For spherically symmetric nonrotating stars, modes are fully separable in the spherical coordinates and the WKB approximation can be directly applied to the Ordinary Differential Equation governing the radial part of the eigenfunction. Consequently, the solutions can be found in closed form and the requirement that they match the boundary conditions yields the well-known asymptotic formulae. In rotating stars, modes are not separable in the latitudinal and radial directions and the WKB approximation is applied to the full 3D perturbation equations. At the leading order, one obtains the eikonal equation $\omega^2 = c_s^2 k^2 + \omega_c^2$, where $k = \nabla\Psi$ is the wave vector at the point x, c_s is the sound velocity and ω_c is the cut-off frequency. The acoustic ray model then consists in looking for solutions of this equation along a given path, called the ray, which is tangent to the wave vector. Moreover, the ray evolution can be described by Hamilton's equations where the Hamiltonian is the wave frequency ω. This procedure is similar to the short-wavelength limit of electromagnetic waves which leads to geometrical optics or to the $\hbar \rightarrow 0$ limit of quantum physics which leads to classical mechanics. Then, in all these cases, the main issue is to find mode solutions from waves propagating along the rays and constructively interfering. This issue, which has been first investigated in the context of quantum physics, depends on the nature of the dynamics. While the integrable case is well understood since the works of Einstein (1917), Brillouin (1926) and Keller (1958), the non-integrable case has been an active field of research in the last thirty years under the name of quantum chaos (Gutzwiller 1990), (Ott 1993).

We have studied the non-integrable dynamics of acoustic rays in rapidly rotating polytropic stars and took advantage of the concepts and methods developed in quantum chaos to interpret the ray dynamics in terms of p-modes properties. This work has been summarized in Lignières & Georgeot (2008) and will be detailed and extended in a forthcoming paper

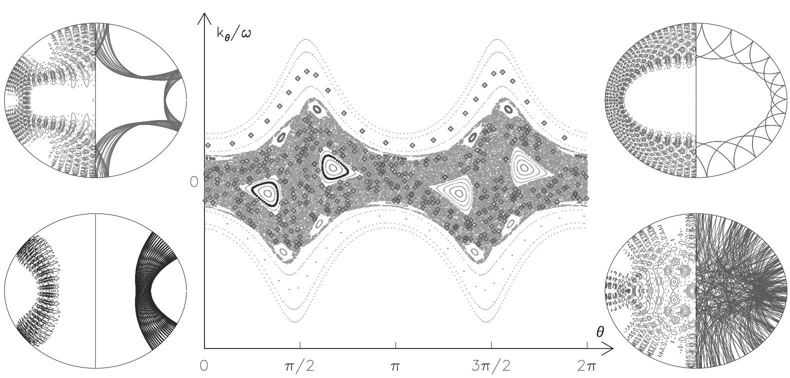

Figure 1: (Color online) Poincaré Surface of Section, typical acoustic rays and modes for a $\Omega = 0.59\Omega_K$ rotating polytropic star of index 3. A 6-period island ray (light grey/magenta), a 2-period island ray (dark grey/blue), a whispering gallery ray (light grey/green) and a chaotic ray (grey/red) are shown together with their imprint on the Poincaré Surface of Section (diamonds in the center figure). The axisymmetric mode distributions are shown on a meridional plane through the level curves, full (resp. dashed) lines corresponding to positive (resp. negative) values.

(Lignières & Georgeot 2009). We found that the asymptotic p-modes spectrum can be described as a superposition of regular and irregular frequency subsets respectively associated with near-integrable and chaotic regions of the ray dynamics phase space. A spectrum is said to be regular if it can be described by a function of N integers, N being the number of degree of freedom of the system (here $N = 2$ for a given azimuthal number m). This is illustrated in Fig. 1 for the case of a $\Omega = 0.59\Omega_K$ rotating polytropic model of index 3. According to the Poincaré Surface of Section which visualizes the phase space structure, there are three near-integrable regions and a large central chaotic region. To test the prediction of the asymptotic theory, we computed high-frequency modes of the same stellar model and build phase-space representations of these modes to associate them with phase space regions. In Fig. 1 modes associated with the four main regions of the phase space, namely the 2-period island chain, the central chaotic region, the 6-period island chain and the whispering gallery region, are shown. As a result, we verified that the mode sub-sets associated with a three near-integrable regions display regularities while the mode sub-set associated with the chaotic region is irregular. Other aspects of the asymptotic theory, in particular quantitative predictions on the regularities and on the statistical properties of the irregular frequency subset, are discussed in Lignières & Georgeot (2008) and Lignières & Georgeot (2009).

Here we shall focus on estimating the visibility of these different classes of modes to determine which modes are likely to be observed. This is crucial information if one wants to use the asymptotic theory to analyze the observed spectrum.

Mode visibility

For spherical stars, the disk-integrated cancellation effects increase rapidly with the degree of the spherical harmonic, thus enabling us to discard large degree modes (say $\ell \geq 5$) for mode identification. We have determined this effect for high frequency axisymmetric p-modes of a $\Omega = 0.59\Omega_K$ rotating polytropic star by computing their disk-averaging factor:

$$D(i) = \frac{1}{\pi R_e^2 \delta T_0} \iint_{S_v} \delta T(\theta, \phi) dS \cdot e_i \quad (1)$$

where i is the inclination angle between the line-of-sight and the rotation axis, e_i is a unit vector in the observer's direction and δT is the spatial part of the Lagrangian temperature perturbation at the stellar surface. The mode amplitude is normalized by δT_0, the root mean square of the perturbation over the whole stellar surface S

$$\delta T_0 = \left(\iint_S \delta T^2(\theta, \phi) dS \right)^{1/2} \quad (2)$$

and the visible surface S_v has been normalized by πR_e^2, the visible surface of a star seen pole-on. With such a normalization the disk-averaging factor of a uniformly distributed surface amplitude seen pole-on is unity. The method used to calculate these integrals will be detailed in Lignières & Georgeot (2009).

Figure 2 shows the spectrum of axisymmetric modes whose disk-averaging factor exceeds 2.5 percent, in the high frequency range $[9\omega_1, 12\omega_1]$ where ω_1 is the lowest acoustic frequency. A first result is that the disk-averaging effect does not allow as many modes to be discarded as for spherical stars. In a given frequency interval and for the same visibility threshold, we find that the number of visible modes is more than three time higher in the $\Omega = 0.59\Omega_K$ star than in a spherical star. A second result is that the 2-period island modes and the chaotic modes have similar visibilities and both types of modes are significantly more visible than the 6-period island modes and whispering gallery modes. This is not surprising in the case of the 2-period island modes as their horizontal wavelength is small compared to that of the other

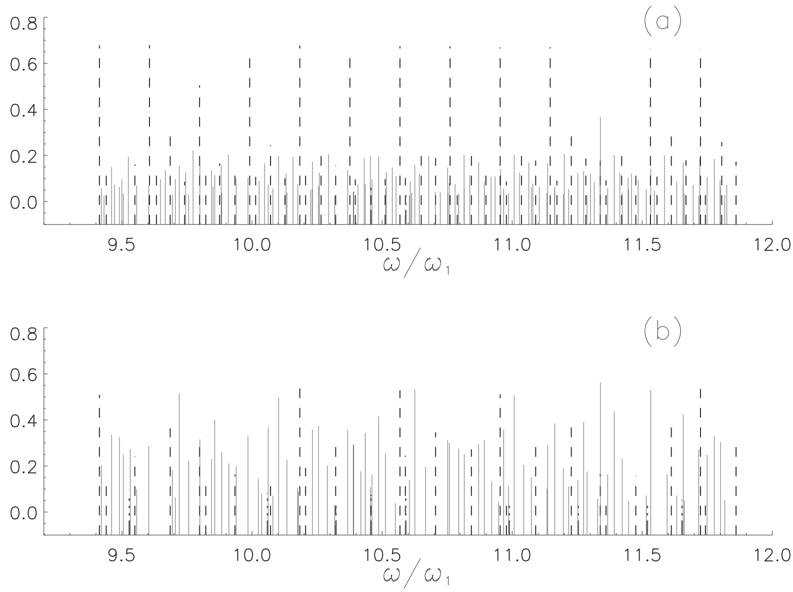

Figure 2: (Color online) Frequency spectrum of axisymmetric modes where the amplitude is given by the normalized disk-averaging factor $D(i)$ (see Eq. (1)) for a star seen pole-on (a) and equator-on (b). Only frequencies such that $D(i) \geq 2.5\%$ are displayed and antisymmetric modes fully cancel out equator-on. The 2-period island modes (dashed lines/blue) and the chaotic modes (continuous lines/red) are the most visible while only a few 6-period island modes (dotted-dashed lines/magenta) and none whispering gallery mode exceed the 2.5% level.

modes. However, on this basis, the chaotic modes should be less visible than the 2-period island modes. We think the explanation is that the irregular nature of the node pattern of the chaotic modes makes the cancellation less effective than it is for regularly spaced nodes. Some 6-period island modes are visible while no whispering gallery modes exceed the chosen threshold. Nevertheless, even the whispering gallery modes cannot be strictly discarded as they may undergo an avoided crossing with a more visible mode.

Conclusion

These results suggest that in the observed spectra of rapidly rotating stars, all high frequency p-modes do not follow a regular pattern and that such patterns might be hidden within an irregular spectrum. In this context, it would be interesting to design methods to extract regular patterns from spectra like the ones shown in Fig. 2.

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Some more LOC announcements to come.