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A New Elliptic Mean

By

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Dedicated to Professor Janusz Matkowski on the occasion of his sixtieth birthday

Abstract

A homogeneous mean is described which arises in a geometric context (circle isoperimetric to an ellipse). With the aid of the index function of a mean it is shown that the considered mean is subadditive.

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1. Introduction

A function M: $(0,\infty)^2 \to (0,\infty)$ is said to be a *mean* if

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad a,b > 0.$$

A mean M is called *strict* if for all $a,b > 0$, $a \neq b$, these inequalities are sharp, *symmetric* if $M(a,b) = M(b,a)$ for all $a,b > 0$, and *homogeneous* if $M(ta,tb) = tM(b,a)$ for all $a,b,t > 0$.

In the geometric context of ellipses, one can obtain, in a natural way, some functions of two variables which are means. For instance:

the problem of finding the radius r of a circle which has the same area as a given ellipse, with semi-axes a and b, leads to the geometric mean:

$$r = G(a,b) := \sqrt{ab}, \quad a,b > 0;$$

(circle and ellipse will have the same area $S = \pi r^2 = \pi ab$);

the problem of determining the distance between the center of an ellipse and a point on the curve leads to a one-parameter family of elliptic means [5].

The present note demonstrates that the problem of finding the radius r of a circle which has the same perimeter as a given ellipse, with semi-axes a and b, leads to an apparently new elliptic mean:

$$E(a,b) := \begin{cases} \frac{2}{\pi} a \mathbb{E}\left(\sqrt{1-\frac{b^2}{a^2}}\right), & a \geq b > 0 \\ \frac{2}{\pi} b \mathbb{E}\left(\sqrt{1-\frac{a^2}{b^2}}\right), & b \geq a > 0, \end{cases} \tag{1}$$

where $\mathbb{E}$ denotes the complete elliptic integral of the second kind

$$\mathbb{E}(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2(\sin\theta)^2}\, d\theta, \quad k \in [-1,1]; \tag{2}$$

a circle of radius $r = E(a,b)$ and the ellipse will have the same perimeter

$$P = 2\pi E(a,b).$$

Although it has long been known how to determine the perimeter of an ellipse [in fact, the well-known integral (2) arose in the rectification of ellipses], it was apparently not recognized that the symmetric expression (1) may be interpreted as a homogeneous mean (cf. [1], [2]).

From the geometric context, it is intuitively clear that E is a mean since, say for $b \leq a$, there holds the following relation between the perimeter P of the ellipse and of the inscribed and circumscribed circle:

$$2\pi b \leq P \leq 2\pi a,$$

hence $b \leq E(a,b) \leq a$.

In this note, applying the notion of the *index function of a mean*, it is shown that E is a homogeneous, strict, symmetric and, which seems to be interesting and not intuitively obvious, subadditive mean. Some relations to the hypergeometric function and Gauss-Kummer series are mentioned.

2. Result and Proof

For an arbitrary function $M\colon (0,\infty)^2 \to (0,\infty)$, one can define a function $f_M\colon (-1,1) \to \mathbb{R}$ by

$$f_M(t) := M(1+t, 1-t), \quad -1 < t < 1.$$

If M is a homogeneous mean the function f_M is called the *index function of M*.

Let A denote the arithmetic mean $A(a,b) := \frac{a+b}{2}$.

We need the following (cf. [4] for more information about index functions).

Lemma 1. *Let* $M\colon (0,\infty)^2 \to (0,\infty)$ *be a function. Then*

1. *M is a homogeneous mean iff*

$$M(a,b) = A(a,b) f_M\left(\frac{a-b}{a+b}\right), \quad a,b > 0,$$

and

$$1 - |t| \le f_M(t) \le 1 + |t|, \quad -1 < t < 1;$$

2. *if M is a homogeneous mean, then*

(*a*) *M is symmetric iff* f_M *is even,*
(*b*) *M is strict iff*

$$1 - |t| < f_M(t) < 1 + |t|, \quad -1 < t < 1,$$

(*c*) *M is subadditive iff* f_M *is convex.*

We prove the following

Theorem 1. *The function E defined by* (1) *is a strict, symmetric and homogeneous mean. Moreover, it can be decomposed into*

$$E(a,b) = A(a,b) f_E\left(\frac{a-b}{a+b}\right), \quad a,b > 0, \tag{3}$$

where $A(a,b) = \frac{a+b}{2}$ *is the arithmetic mean, the function* $f_E\colon (-1,1) \to (0,2)$, *given by*

$$f_E(t) := \frac{2}{\pi}(1+|t|)\mathbb{E}\left(2\frac{\sqrt{|t|}}{1+|t|}\right), \quad -1 < t < 1, \tag{4}$$

is the index function of E, and $t := \frac{a-b}{a+b}$ *is a shape parameter* $(-1 < t < 1)$. *Furthermore, E is subadditive, that is*

$$E(a_1 + a_2, b_1 + b_2) \le E(a_1, b_1) + E(a_2, b_2), \quad a_1, a_2, b_1, b_2 > 0.$$

Proof. The definition (1) of the mean E may also be written in this way:

$$E(a,b) := \begin{cases} \frac{2}{\pi} a\mathbb{E}\Big(2\frac{\sqrt{t}}{1+t}\Big), & a \geq b > 0, \quad 0 \leq t < 1 \\ \frac{2}{\pi} b\mathbb{E}\Big(2\frac{\sqrt{-t}}{1-t}\Big), & b \geq a > 0, \quad -1 < t \leq 0, \end{cases}$$

where $t = (a-b)/(a+b)$, or equivalently

$$E(a,b) := \frac{2}{\pi}\max(a,b)\mathbb{E}\left(2\frac{\sqrt{|t|}}{1+|t|}\right), \quad a,b > 0, \quad t \in (-1,1).$$

Following [4], we have the decomposition

$$\max(a,b) = A(a,b)(1+|t|), \quad a,b > 0, \quad t \in (-1,1),$$

leading to the representation

$$E(a,b) := \frac{2}{\pi}A(a,b)(1+|t|)\mathbb{E}\left(2\frac{\sqrt{|t|}}{1+|t|}\right), \quad a,b > 0, \quad t \in (-1,1),$$

from which follows (3) and (4). By (2),

$$\mathbb{E}\left(2\frac{\sqrt{|t|}}{1+|t|}\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(2\frac{\sqrt{|t|}}{1+|t|}\right)^2 (\sin\theta)^2}\, d\theta, \quad t \in (-1,1),$$

and, for all $\theta \in (0, \frac{\pi}{2})$,

$$\left(\frac{1-|t|}{1+|t|}\right)^2 = 1 - \left(2\frac{\sqrt{|t|}}{1+|t|}\right)^2 < 1 - \left(2\frac{\sqrt{|t|}}{1+|t|}\right)^2 (\sin\theta)^2 < 1.$$

Hence, making use of (4), we get

$$1 - |t| < f_E(t) < 1 + |t|, \quad -1 < t < 1.$$

Moreover, for all $t \in [0,1)$ we have

$$f_E''(t) = \frac{2}{\pi}(1+|t|)\int_0^{\frac{\pi}{2}} \frac{4(\sin\theta)^2[(3t^2-6t-1)(\sin\theta)^2 - t^3 - 3t + 2]}{(t+1)^3[-4t(\sin\theta)^2 + t^2 + 2t + 1]^{3/2}}\, d\theta.$$

Since, for all $t \in [0,1)$ and $\theta \in (0, \frac{\pi}{2})$,

$$\begin{aligned}(3t^2-6t-1)(\sin\theta)^2 - t^3 - 3t + 2 &> (3t^2-6t-1) - t^3 - 3t + 2 \\ &= (1-t)(1-t)^2 > 0,\end{aligned}$$

and

$$-4t(\sin\theta)^2 + t^2 + 2t + 1 > (-4t) + t^2 + 2t + 1 = (1-t)^2 > 0,$$

we infer that $f_E'' > 0$ in [0,1). Consequently, f_E is convex in [0,1). Since f_E is an even function, it must be convex in $(-1,1)$. The theorem follows by Lemma 1. □

3. Remarks

Remark 1. *To show that E is a strict mean we can use the following direct argument. Take* $a, b > 0$. *Without any loss of generality we may assume that* $a \geq b$. *The inequalities* $0 = \sin 0 < \sin\theta < \sin\frac{\pi}{2}$ *for all* $\theta \in (0, \frac{\pi}{2})$ *imply that*

$$b = a\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\left(\sin\frac{\pi}{2}\right)^2} < a\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)(\sin\theta)^2}$$
$$< a\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)(\sin 0)^2} = a.$$

Hence we get

$$b < \frac{2}{\pi}\int_0^{\frac{\pi}{2}} a\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)(\sin\theta)^2}\,d\theta < a,$$

which, according to the definition of E, gives

$$\min(a, b) = b < E(a, b) < a = \max(a, b),$$

and proves that E is a strict mean. Symmetry and homogeneity of E are obvious.

Remark 2. *Using the well-known representation of the complete elliptic integral of the second kind as a hypergeometric function,*

$$\mathbb{E}(k) := \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad k \in (-1, 1),$$

the index function of the mean E can be written in the following way:

$$f_E(t) = (1 + |t|)\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{4|t|}{(1 + |t|)^2}\right), \quad t \in (-1, 1).$$

Applying Kummer's quadratic transformation of the hypergeometric function, there results the concise representation

$$f_E(t) = {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; t^2\right), \quad t \in (-1, 1),$$

whose explicit form is known as the (rapidly converging) Gauss-Kummer series

$$f_E(t) = \sum_{n=0}^{\infty} \binom{1/2}{n}^2 t^{2n} = 1 + \frac{t^2}{4} + \frac{t^4}{64} + \frac{t^6}{256} + \cdots, \quad t \in (-1, 1).$$

Recalling that

$$P = 2\pi r = 2\pi E(a, b) = 2\pi \frac{a+b}{2} f_E(t) = \pi(a+b) f_E(t),$$

the corresponding explicit form for the perimeter P of an ellipse, with semi-axes a and b, is

$$P = \pi(a+b)\left(1 + \frac{t^2}{4} + \frac{t^4}{64} + \frac{t^6}{256} + \cdots\right), \quad a, b > 0, \ t := \frac{a-b}{a+b},$$

which is stated (without derivation) in almost any technical handbook (cf. e.g. [3]*). Geometric interpretation: the perimeter of an ellipse is the elliptic mean of the perimeters of inscribed and circumscribed circle.*

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