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Some Relations Between Generalized Fibonacci and Catalan Numbers

By

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Abstract

In a recent paper Aleksandar Cvetković, Predrag Rajković and Miloš Ivković proved that for the Catalan numbers C_n the Hankel determinants of the sequence $C_n + C_{n+1}$ are Fibonacci numbers. Their proof depends on special properties of the corresponding orthogonal polynomials. In this paper we give a generalization of their result by other methods in order to give more insight into the situation.

Mathematics Subject Classifications: 05A10, 05A19, 05A30, 11Bxx, 15A15.
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1. A Survey of Known Results

The classical Fibonacci polynomials $F_n(x,s) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} \times x^{n-1-2k} s^k$ are intimately related to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$: The Fibonacci polynomials $F_n(x,1)$, $n > 0$, are a basis of the vector space of polynomials. If we define the linear functional L by $L(F_{n+1}) = \delta_{n,0}$ then we get $L(x^{2n+1}) = 0$ and $L(x^{2n}) = (-1)^n C_n$.

We will now sketch how this fact can be generalized.

Given any sequence $t = (t(n))_{n=0}^{\infty}$ of positive real numbers we define the t-Fibonacci polynomials (cf. [5]) by

$$F_n(x,t) = xF_{n-1}(x,t) + t(n-3)F_{n-2}(x,t) \tag{1.1}$$

with initial conditions $F_0(x,t) = 0$, $F_1(x,t) = 1$.

If s is a real or complex number and $t(n) = s$ for all $n \in \mathbb{N}$ this reduces to the classical Fibonacci polynomials $F_n(x,s)$ introduced above.

The first terms are

$$F_2(x,t) = x, \quad F_3(x,t) = x^2 + t(0), \quad F_4(x,t) = x^3 + xt(0) + xt(1), \ldots$$

We state for later purposes the recurrences for the subsequences with even or odd indices.

$$\begin{aligned} F_{2n}(x,t) = (x^2 + t(2n-4) + t(2n-3))F_{2n-2}(x,t) - \\ - t(2n-4)t(2n-3)F_{2n-4}(x,t) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} F_{2n+1}(x,t) = (x^2 + t(2n-3) + t(2n-2))F_{2n-1}(x,t) - \\ - t(2n-3)t(2n-4)F_{2n-3}(x,t). \end{aligned} \tag{1.3}$$

The polynomials $F_n(x,t)$, $n > 0$, are a basis of the vector space $\mathbb{P}$ of all polynomials in x. We can therefore define a linear functional L on $\mathbb{P}$ by

$$L(F_n) = \delta_{n,1}. \tag{1.4}$$

Let $\hat{F}_n(x,t) = \frac{F_n(x,t)}{t(0)t(1)\cdots t(n-2)}$.

Then we have

$$x\hat{F}_n = t(n-1)\hat{F}_{n+1} - \hat{F}_{n-1}.$$

Define now the numbers $a_{n,k} = (-1)^{\lceil \frac{n+k}{2} \rceil} L(x^n \hat{F}_{k+1})$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x.

They satisfy

$$\begin{aligned} a_{0,k} &= \delta_{0,k} \\ a_{n,k} &= a_{n-1,k-1} + t(k)a_{n-1,k+1} \end{aligned} \tag{1.5}$$

where, $a_{n,k} = 0$ if $k < 0$.

They have an obvious combinatorial interpretation. Consider all nonnegative lattice paths in $\mathbb{R}^2$ which start in $(0,0)$ with upward steps $(1,1)$ und downward steps $(1,-1)$. We associate to each upward step ending on the height k the weight 1 and to each downward step ending on the height k the weight $t(k)$. The weight of the path is the product of the weights of all steps of the path. Then $a_{n,k}$ is the weight of all nonnegative lattice paths from $(0,0)$ to (n,k).

It is clear that $a_{2n+1,0} = 0$. If we set $a_{2n,0} = C_n(t)$, then $C_n(t)$, which we call a t-Catalan number (cf. [4]), is an analogue of the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, because it is well known that the number of such paths equals C_n.

It is easy to give a recurrence for these t-Catalan numbers. To this end decompose each lattice path from $(0,0)$ to $(2n,0)$ into the first path which returns to the x-axis and the rest path. The first path goes from $(0,0)$ to $(2k+2,0)$, $0 \le k \le n-1$, and consists of a rising segment followed by a path from $(0,0)$ to $(2k,0)$ (but one level higher) and a falling segment.

Thus

$$C_n(t) = t(0)\sum_{k=0}^{n-1} C_k(Et)C_{n-k-1}(t), \quad C_0(t) = 1. \tag{1.6}$$

Here Et denotes the shifted sequence $Et = (t(1), t(2), \ldots)$.

This is an analogue of the recursion

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \quad C_0 = 1,$$

for the classical Catalan numbers.

Since

$$L(x^{2n+1}) = 0 \quad \text{and} \quad L(x^{2n}) = (-1)^n C_n(t) \text{ we get}$$

Theorem 1. *Let L be the linear functional on the vector space $\mathbb{P}$ of all polynomials defined by* (1.4). *This can be characterized by*

$$L(x^{2n}) = (-1)^n C_n(t), \qquad L(x^{2n+1}) = 0 \tag{1.7}$$

for all $n \in \mathbb{N}$.

This theorem (cf. [5]) gives a connection between t-Fibonacci polynomials and t-Catalan numbers. The purpose of this paper is to show another connection between these numbers.

First we propose an elementary method using LU-factorization (cf. e.g. [8], where also other methods for treating Hankel determinants are listed). To this end we generalize two well-known triangular Catalan matrices (cf. e.g. [1, 4, 9]):

Let

$$c(n,k) = a_{2n,2k}. \tag{1.8}$$

This gives $c(n,0) = C_n(t)$.

Then it is clear that the following recursion holds:

$$c(0,k) = [k=0]$$
$$c(n,0) = t(0)c(n-1,0) + t(0)t(1)c(n-1,1), \quad n>0,$$
$$c(n,k) = c(n-1,k-1) + (t(2k-1)+t(2k))c(n-1,k) + \\ + t(2k)t(2k+1)c(n-1,k+1), \quad n>0, \quad k>0. \tag{1.9}$$

We may interpret $c(n,k)$ as the weight of another sort of nonnegative lattice paths from $(0,0)$ to (n,k) with upward, downward and horizontal steps, where each upward step has weight 1, each downward step ending at height k has weight $t(2k)t(2k+1)$ and each horizontal step in height k has weight $t(2k-1)+t(2k)$. Setting

$$cc(n,k) = \sqrt{t(0)t(1)\cdots t(2k-1)}\,c(n,k) \tag{1.10}$$

we get another weight on these lattice paths which is symmetric, i.e. upward steps and downward steps between the same heights have the same weight. Furthermore we have $cc(n,0) = c(n,0) = C_n(t)$. They satisfy

$$cc(0,k) = [k=0]$$
$$cc(n,0) = t(0)cc(n-1,0) + \sqrt{t(0)t(1)}c(n-1,1), \quad n>0,$$
$$cc(n,k) = \sqrt{t(2k-2)t(2k-1)}cc(n-1,k-1) + \\ + (t(2k-1)+t(2k))cc(n-1,k) + \\ + \sqrt{t(2k)t(2k+1)}c(n-1,k+1), \quad n>0, \quad k>0. \tag{1.11}$$

If we decompose a lattice path from $(0,0)$ to $(m+n,0)$ into a path from $(0,0)$ to (m,k) and a second path from (m,k) to $(m+n,0)$, then the weight of the second path is identical with $cc(n,k)$ because of the symmetry. This gives the identity

$$\sum_{k\geq 0} cc(m,k)cc(n,k) = cc(m+n,0) \tag{1.12}$$

or equivalently

$$\sum_{k\geq 0} c(m,k)c(n,k)t(0)t(1)\cdots t(2k-1) = c(m+n,0) = C_{m+n}(t).$$

This may be interpreted in the following form:

Consider the triangular matrices

$$P_n = (cc(i,j))_{i,j=0}^{n-1}. \tag{1.13}$$

Then

$$P_n P_n^t = (C_{i+j}(t))_{i,j=0}^{n-1}. \tag{1.14}$$

This leads immediately to the determinant of the Hankel matrix

$$\det (C_{i+j}(t))_{i,j=0}^{n-1} = \det P_n P_n^t = \prod_{k=0}^{n-1} t(0)t(1)\cdots t(2k-1) \tag{1.15}$$

For the second Catalan matrix let

$$d(n,k) = a_{2n+1,2k+1}. \tag{1.16}$$

Then

$$d(n,0) = a_{2n+1,1} = a_{2n+2,0} = C_{n+1}(t). \tag{1.17}$$

The recursion takes now the following form:

$$\begin{aligned} d(0,k) &= t(0)[k=0] \\ d(n,0) &= (t(0)+t(1))d(n-1,0) + t(1)t(2)d(n-1,1), \quad n>0, \\ d(n,k) &= d(n-1,k-1) + (t(2k)+t(2k+1))d(n-1,k) + \\ &\qquad + t(2k+1)t(2k+2)d(n-1,k+1), \quad n>0, \quad k>0. \end{aligned} \tag{1.18}$$

Setting

$$dd(n,k) = \frac{\sqrt{t(1)\cdots t(2k)}}{t(0)} d(n,k) \tag{1.19}$$

we get another weight on these lattice paths which is symmetric and satisfies $dd(0,k) = [k=0]$.

This gives the identity

$$\sum_{k\geq 0} dd(m,k)dd(n,k) = dd(m+n,0) = C_{m+n+1}(t) \tag{1.20}$$

or equivalently

$$\sum_{k\geq 0} d(m,k)d(n,k)t(1)\cdots t(2k) = t(0)d(m+n,0) = t(0)C_{m+n+1}(t).$$

This result may be interpreted in the following form:

Consider the triangular matrices

$$Q_n = (dd(i,j))_{i,j=0}^{n-1}. \tag{1.21}$$

Then

$$Q_n Q_n^t = (C_{i+j+1}(t))_{i,j=0}^{n-1} \tag{1.22}$$

and we get the determinant of the second Hankel matrix

$$\det(C_{i+j+1}(t))_{i,j=0}^{n-1} = \det Q_n Q_n^t = \prod_{k=0}^{n} t(0)t(1)t(2)\cdots t(2k-2). \tag{1.23}$$

Remark. This implies that

$$\det Q_n Q_n^t = t(0)t(2)\cdots t(2n-2)\det P_n P_n^t. \tag{1.24}$$

The most important special cases are the following:

1) $t(n) = q^n s$: The *Carlitz q-Fibonacci polynomials* and the *Carlitz q-Catalan numbers.*
Here we get ([2–4])

$$\prod_{k=0}^{n-1} t(0)t(1)\cdots t(2k-1) = q^{\frac{(n-1)n(4n-5)}{6}} s^{2\binom{n}{2}} = q^{2\sum_{i=0}^{n-1} i^2 - \binom{n}{2}} s^{2\binom{n}{2}}$$

and

$$\prod_{k=0}^{n} t(0)t(1)\cdots t(2k-2) = q^{\frac{n(n-1)(4n+1)}{6}} s^{n^2} = q^{2\sum_{i=0}^{n-1} i^2 + \binom{n}{2}} s^{n^2}.$$

It should be noted that in this case

$$F_n(x,s,q) = \sum_{k=0}^{n-1} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{2\binom{k}{2}} s^k x^{n-2k-1}.$$

2) Another interesting special case arises for

$$t(2k) = rq^k, \quad t(2k+1) = sq^k,$$

where r and s are positive real numbers.
The corresponding Fibonacci polynomials are

$$0, 1, x, x^2 + r, x^3 + rx + sx, x^4 + rx^2 + qrx^2 + sx^2 + qr^2, \ldots.$$

The corresponding Catalan numbers $C_n(q,r,s)$ (the *Polya-Gessel q-Catalan numbers*) are

$$1, r, r^2 + rs, r^3 + 2r^2 s + qr^2 s + rs^2, \ldots.$$

They satisfy the recurrence (cf. [4, 7])

$$C_{n+1}(q,r,s) = rC_n(q,r,s) + s\sum_{k=0}^{n-1} q^k C_k(q,r,s)C_{n-k}(q,r,s).$$

In this case we have ([4])

$$\prod_{k=0}^{n-1} t(0)t(1)\cdots t(2k-1) = q^{2\binom{n}{3}}(rs)^{\binom{n}{2}}$$

and

$$\prod_{k=0}^{n} t(0)t(1)\cdots t(2k-2) = q^{\sum_{k=0}^{n-1} k^2} r^{\binom{n+1}{2}} s^{\binom{n}{2}}.$$

2. The Main Theorem

Recently it has been proved ([6]) that for the sequence $a(n) = \frac{1}{n+1}\binom{2n}{n} + \frac{1}{n+2}\binom{2n+2}{n+1} = C_n + C_{n+1}$ the Hankel determinants are explicitly given by $h_0(n) = F_{2n+1}$ and $h_1(n) = F_{2n+2}$ with Fibonacci numbers F_n. We will now generalize these results:

Theorem 2. *Consider the sequence*

$$a(n,z,t) = C_n(t) + zC_{n+1}(t). \tag{2.1}$$

Let

$$h_0(n,z,t) = (a(i+j,z,t))_{i,j=0}^{n-1} \tag{2.2}$$

and

$$h_1(n,z,t) = (a(i+j+1,z,t))_{i,j=0}^{n-1} \tag{2.3}$$

be the associated Hankel matrices.

Then we have

$$\det h_0(n,z,t) = F_{2n+1}(1,zt)\prod_{k=0}^{n-1} t(0)t(1)\cdots t(2k-1) \tag{2.4}$$

and

$$\det h_1(n,z,t) = F_{2n+2}(1,zt)\prod_{k=0}^{n} t(0)t(1)\cdots t(2k-2). \tag{2.5}$$

To prove this observe the following fact (cf. [1]):
The matrix $R_n := (cc(i+1,j))_{i,j=0}^{n-1}$ satisfies $R_n = P_n J_n$, where

$$J_n = \begin{pmatrix} s_0 & v_0 & 0 & \cdots & 0 \\ v_0 & s_1 & v_1 & \cdots & 0 \\ 0 & v_1 & s_2 & \cdots & 0 \\ & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & s_{n-1} \end{pmatrix} \tag{2.6}$$

with $s_0 = t(0)$, $s_k = t(2k-1) + t(2k)$, $k > 0$, and

$$v_i = \sqrt{t(2i)t(2i+1)}.$$

Furthermore we have

$$h_1(n,0,t) = (C_{i+j+1}(t))_{i,j=0}^{n-1} = R_n P_n^t = P_n J_n P_n^t.$$

Now it is easily verified that $\det J_n = t(0)t(2)\cdots t(2n-2)$. We have thus another proof of (1.24).

Now we have

$$h_0(n,z,t) = P_n P_n^t + z P_n J_n P_n^t = P_n(I + zJ_n)P_n^t. \tag{2.7}$$

We must now compute

$$d(n) = \det(I + zJ_n). \tag{2.8}$$

If we expand with respect to the last row we get the recursion

$$d(n) = (1 + z(t(2n-3) + t(2n-2))d(n-1) - \\ - z^2 t(2n-3)t(2n-4)d(n-2) \tag{2.9}$$

with initial values

$$d(1) = 1 + zt(0), \quad d(2) = 1 + z(t(0) + t(1) + t(2)) + z^2 t(0)t(2)$$

Therefore we get

$$d(n) = F_{2n+1}(1, zt). \tag{2.10}$$

In the same way we may obtain the second Hankel determinant.

Finally we note another interesting Hankel determinant.

Theorem 3. *Set*

$$b(2n,z,t) = C_n(t) + zC_{n+1}(t), \\ b(2n+1,z,t) = 0. \tag{2.11}$$

Define the Hankel matrices

$$k_0(n,z,t) = (b(i+j,z,t))_{i,j=0}^{n-1} \tag{2.12}$$

and

$$k_1(n,z,t) = (b(i+j+1,z,t))_{i,j=0}^{n-1}. \tag{2.13}$$

Then we get

$$\det k_0(n,z,t) = t(0)^{n-1} t(1)^{n-2} \cdots t(n-1) F_{n+1}(1,zt) F_{n+2}(1,zt) \tag{2.14}$$

and

$$\det k_1(2n,z,t) = (-1)^n t(0)^{2n} t(1)^{2n-2} t(2)^{2n-2} \cdots t(2n-3)^2 t(2n-2)^2 F_{2n+2}(1,zt)^2. \tag{2.15}$$

The determinants with odd index are 0.

The proof follows immediately from Theorem 2. To this end write the rows and columns with even index first and then the others in their natural order. Then the first matrix splits into one of the form $h_0(k,z,t)$ and one of the form $h_1(l,z,t)$. For $n = 2m$ the second matrix splits into two matrices $h_1(m,z,t)$. For n odd one of the two matrices has a row of zeroes.

3. Another Method of Proof

After completion of this paper Christian Krattenthaler has remarked that some of our results become almost trivial modulo a theorem of Lindström-Gessel-Viennot on non-overlapping lattice paths (see [10] for a detailed account of this theorem). We now sketch this approach.

Theorem (Lindström-Gessel-Viennot). *Given initial points A_0, $A_1, \ldots, A_{n-1}$ and endpoints $E_0, E_1, \ldots, E_{n-1}$. Then*

$$\det(P(A_i \to E_j)) = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \cdot P^+(\mathbf{A} \to \mathbf{E}_\sigma),$$

where $P(A_i \to E_j)$ denotes the weight of all admissible lattice paths from A_i to E_j and $P^+(\mathbf{A} \to \mathbf{E}_\sigma)$ denotes the weight of all families $(P_0, P_1, \ldots, P_{n-1})$ of nonoverlapping lattice paths such that P_i goes from A_i to $E_{\sigma(i)}$ for $i = 0, 1, \ldots, n-1$.

If you choose $A_i = (-2i, 0)$ and $E_i = (2i, 0)$, $i = 0, 1, \ldots, n-1$, then $P(A_i \to E_j) = C_{i+j}(t)$. In this case it is evident (by induction) that the only possible σ is the identity and $P_i : A_i \to E_i$ is the (highest possible) path which goes $2i$ steps upwards to the point $(0, 2i)$ and then downwards to $E_i = (2i, 0)$. The weight of this family of paths is evidently given by (1.15).

If you choose $A_i = (-2i, 0)$ and $E_i = (2i+2, 0)$, $i = 0, 1, \ldots, n-1$, then $P(A_i \to E_j) = C_{i+j+1}(t)$. In this case it is also evident (by induction) that the only possible σ is the identity and $P_i : A_i \to E_i$ is the (highest possible) path which goes $2i+1$ steps upwards to the point $(0, 2i+1)$ and then downwards to $E_i = (2i+2, 0)$. The weight of this family of paths is evidently given by (1.23).

For the Hankel determinant $\det(C_{i+j}(t) + zC_{i+j+1}(t))$ we use the linearity in the columns to write it as a sum of 2^n simpler determinants. Most of these determinants are 0, because two adjacent columns are proportional. What remains is the sum

$$\det(C_{i+j}(t) + zC_{i+j+1}(t)) = \sum_{k=0}^{n} B_{n,k} z^k, \tag{3.1}$$

where

$$B_{n,k} = \det(C_i(t)C_{i+1}(t) \cdots C_{i+n-k-1}(t)C_{i+n-k+1}(t) \cdots C_{i+n}(t)).$$

For the determinant $\det(C_{i+j+1}(t) + zC_{i+j+2}(t))$ we get in the same way

$$\det(C_{i+j+1}(t) + zC_{i+j+2}(t)) = \sum_{k=0}^{n} D_{n,k} z^k \tag{3.2}$$

with

$$D_{n,k} = \det(C_{i+1}(t)C_{i+2}(t) \cdots C_{i+n-k}(t)C_{i+n-k+2}(t) \cdots C_{i+n+1}(t)).$$

Now we apply again the Lindström-Gessel-Viennot theorem. To obtain $B_{n,k}$ we choose $A_i = (-2i, 0)$ and $E_j = (2j + 2[j > n-k])$. For $D_{n,k}$ we choose $A_i = (-2i, 0)$ and $E_j = (2j + 2 + 2[j > n-k])$.

It is again evident that in both cases the only possible permutation is the identity. We study first $B_{n,k}$. For each family of nonintersecting paths the path P_i begins with $2i$ upward steps. After this there are two possibilities: Either the next step of P_{n-1} is an upward step, case 1, (in which case P_{n-1} is of maximal height $2n-1$ and the $2n-1$ other steps are downward steps), or the next step of P_{n-1} is a downward step (case 2).

In case 1 the weight of the family is $\left(\prod_{j=0}^{2n-2} t(j)\right) B_{n-1,k-1}$ since there are no restrictions for the other paths. In case 2 the next step of each P_i must be a downward step because the paths are nonintersecting. In this case we replace the first $2i+1$ steps of the path P_i which go from $(-2i, 0)$ to $(1, 2i-1)$ for $i = 1, \ldots, n-1$ by the path which starts in $(-2i+2, 0)$ and has only upward steps ending in $(1, 2i-1)$. The rest of the path remains unchanged. This gives $D_{n-1,k}$. But we have ignored the weight of the downward steps ending in height $1, 3, \ldots, 2n-3$. Thus we get

$$B_{n,k} = \left(\prod_{j=0}^{2n-2} t(j)\right) B_{n-1,k-1} + \left(\prod_{i=0}^{n-2} t(2i+1)\right) D_{n-1,k}. \tag{3.3}$$

The same procedure applied to $D_{n,k}$ gives

$$D_{n,k} = \left(\prod_{j=0}^{2n-1} t(j)\right) D_{n-1,k-1} + \left(\prod_{i=0}^{n-1} t(2i)\right) B_{n,k}. \tag{3.4}$$

Let now

$$B^*_{n,k} = \frac{B_{n,k}}{\prod_{k=0}^{n-1} t(0)t(1)\cdots t(2k-1)} \tag{3.5}$$

and

$$D^*_{n,k} = \frac{D_{n,k}}{\prod_{k=0}^{n} t(0)t(1)\cdots t(2k-2)}. \tag{3.6}$$

Then we get

$$\begin{aligned} B^*_{n,k} &= D^*_{n-1,k} + t(2n-2) B^*_{n-1,k-1} \\ D^*_{n,k} &= B^*_{n,k} + t(2n-1) D^*_{n-1,k-1}. \end{aligned} \tag{3.7}$$

Let now

$$f_{2n} = \sum D^*_{n-1,k} z^k \quad \text{and} \quad f_{2n+1} = \sum B^*_{n,k} z^k.$$

Then we get the recurrence

$$f_n = f_{n-1} + zt(n-3) f_{n-2}.$$

Since $f_3 = F_3(1, zt)$ and $f_4 = F_4(1, zt)$ we see that $f_n = F_n(1, zt)$ for all n. This is equivalent with (2.4) and (2.5).

The t-Fibonacci polynomials can be interpreted as the weight of Morse code sequences (cf. [5]), i.e. sequences of dots (•) which occupy one point and dashes (—) which occupy two points.

Let v be a Morse code sequence on some interval $\{0, 1, \ldots, k-1\}$. If the place $i \in \{0, 1, \ldots, k-1\}$ is occupied by a dot we set $w(i) = x$, if it is the initial point of a dash we set $w(i) = t(i)$. In the other cases let $w(i) = 1$. Now the length of v is k and the weight of v is defined as the product of the weights of all places of the interval, i.e. $w(v) = \prod_{i=0}^{k-1} w(i)$. For example the weight of the sequence –•–••– –• of length 12 is $x^4 t(0)t(3)t(7)t(9)$. The weight of all Morse code sequences with length $n-1$ is given by the t-Fibonacci polynomial $F_n(x, t)$.

Christian Krattenthaler (personal communication) has given a bijective proof of Theorem 2 by associating a Morse code sequence with each family of nonintersecting paths.

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