

TIME EVOLUTION OF LOW FREQUENCY PLASMA MODES

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Abstract

The time evolution of linear, low frequency, plasma waves in the emission region is studied via theoretical properties and numerical solutions of Volterra integral equations of the second kind. Bounds for the growth rates of unstable plasmas, with arbitrary distribution functions, are established. The asymptotic value of the electric field $E(\infty)$ is found by means of the Paley–Wiener theorem for stable plasmas. For initial perturbations of the distribution function $f_1(0)$ which are square integrable, $E(\infty) = 0$ (Landau damping). When $f_1(0)$ contains beams of particles (it is not square integrable) $E(\infty) \neq 0$. The arbitrariness of the time decay of the electric field is obtained straight-forwardly. Some examples of numerical solutions for electron plasma waves and electron Bernstein modes are given.

1. Introduction

The theory of plasma waves has an important field of application in space and planetary sciences. Space plasma is the natural grand laboratory for a variety of undulatory plasma phenomena, over a large spectrum of frequencies (Bauer, this issue). The extensive growth of plasma theory related to fusion physics has perhaps obscured the fact that historically the study of plasma waves began in the domain of cosmic plasmas (Melrose, 1986b).

In the theory of waves we can identify two classes of problems (Drummond, 1961).

1. The analysis of the propagation properties, which may start considering a solution with a fixed real frequency ω , and proceeds then to the calculation of the (complex) wave number $k = k(\omega)$. This is essentially a boundary value problem.
2. The study of the time evolution of a perturbation, which may be performed specifying a real wave number k , and determining then the (complex) frequency $\omega = \omega(k)$. This is a typical initial value problem. The present paper is devoted to problems of the second class: i.e. to obtain the time behaviour of given initial excitations of the plasma.

A perturbed plasma acts as a source of plasma waves. We study the evolution of plasma modes within the emission zone. We assume that this is a large plasma region which has been perturbed at $t = 0$ by local or external agents. The initial excitation of the plasma distribution function, $f_1(\underline{x}, \underline{v}, t = 0)$ (with respect to a pre-existing distribution function $f_0(\underline{v})$ of the homogeneous plasma) is Fourier analyzed in the space variables:

$$f_1(\underline{x}, \underline{v}, 0) = \int \frac{d^3k}{(2\pi)^3} \exp(i\mathbf{k} \cdot \mathbf{x}) f_1(\mathbf{k}, \underline{v}, 0)$$

We examine the evolution in time of each wavenumber component $f_1(\mathbf{k}, \underline{v}, t)$ separately, focussing the attention on the corresponding electric field $\underline{E}(\mathbf{k}, t)$.

There are two classical methods for the study of initial value problems in plasma theory. One was originated by L.D.Landau and is based on Laplace transform techniques (Landau, 1946). The other one was developed by N.G.Van Kampen and K.M.Case and is based on normal modes analysis (Van Kampen, 1955; 1957; 1959). In both approaches important concepts are introduced that have been influential in the development of the theory of plasma waves. However, both methods have some practical limitations. To obtain the solution $E(k, t)$ by Landau's method we must sum the contribution of residues of poles arising from the zeros of the plasma dielectric function $\varepsilon(k, p)$ where $p = -i\omega$. In general, this is a transcendental function whose (infinite) zeros must be searched for numerically in the complex p plane. It has been shown (Ecker and Frömling, 1974) that a large number of poles must be summed to obtain an accurate representation of $E(k, t)$ for finite t when $f_1(k, v, 0)$ has a velocity spread larger than that of the plasma $f_0(v)$. The sum of the residues converges nonuniformly for small t . Even worse is the fact that when the velocity spread of f_1 is small in comparison with that of f_0 the exact solution differs from the sum of residues for all finite t . The development of the normal mode method is limited at the moment to electrostatic waves in a plasma without magnetic field. Since the eigenmodes are singular and the spectrum has continuum and discrete parts, the analysis is delicate and requires great care. Controversial aspects have been debated in the literature (Arthur et al., 1977). Case reformulated some parts of the spectral theory (Case, 1978), while an alternative operator theory has been proposed (Arthur et al., 1977). In practical problems, numerical evaluations are also needed in the normal mode analysis (see e.g. Ecker and Frömling, 1974).

In the present work we propose a third approach for the study of the time behaviour $E(k, t)$, which is based on the theory of Volterra integral equations. It does not suffer from the limitations mentioned above. Some theoretical properties of the solutions can be derived in a simple direct way by this method. We found that it constitutes a useful theoretical complement to the classical approaches, and a convenient replacement when numerical results are required.

2. Purpose of this work

We study the time evolution of plasma wave sources, i.e. the emission region, in the linear approximation, setting the problem in the form of Volterra integral equations of the second

kind. We derive from this approach theoretical properties and numerical results directly in the temporal representation. The equations are of the form

$$\phi(t) + \int_0^t H(t - \tau) \phi(\tau) d\tau = g(t) , \quad (1)$$

where ϕ is the electrostatic potential for modes in the quasi-electrostatic approximation, or may stand for a component of the electric field when the mode is electromagnetic. The kernel $H(t)$ is related to the polarization response function of the particular plasma mode, and depends on the distribution function $f_0(\underline{v})$ of the plasma through a Fourier transform from velocities to time arguments. The r.h.s. term $g(t)$ is the forcing term or datum of the integral equation. In our problem it is related to the initial perturbative distribution $f_1(\underline{x}, \underline{v}, 0)$, again via a Fourier transform from $\underline{v}$ to t . The derivation of the integral equations has been given elsewhere, but in Section 3 we summarize the main formulas. In Section 4 several theoretical properties of the solutions are derived. We call the attention on the simple way that information on the growth bounds of instabilities, or the asymptotic value of the electric field for stable plasmas, can be derived following this method. Section 5 is devoted to a discussion of growth bounds for several low frequency modes in magnetized plasmas with arbitrary distribution functions. Finally, Section 6 shows examples of numerical solutions for electron plasma waves (along the magnetic field) and electron Bernstein modes (normal to the magnetic field).

3. Integral equations for plasma modes

We present here integral equations for linearized $\underline{k}$ mode amplitudes in a homogeneous plasma in the presence of a uniform magnetic field. The kernel H is written for a single species. A sum over species must be used, as convenient for specific problems. For each species, f_0 must be normalized to unity. The expressions for the kernels have been derived in (Gnavi and Gratton, 1987a) using a method based on impulsive perturbations of the plasma particles. Alternatively, H can be computed from the Vlasov kinetic equation by integration along the orbits (Bernstein et al., 1964).

1. Potential modes (quasi-electrostatic approximation). These modes include the plasma resonances, the ion-acoustic and the Bernstein modes. Let $\underline{E}(t) = -i\underline{k}\phi(t)$ (ϕ : electrostatic potential). Then, we obtain Eq. (1) with

$$H(t) = \left(\frac{\omega_p^2}{k^2} \right) \left[k_\perp^2 \left(\frac{\sin \Omega t}{\Omega} \right) + k_\parallel^2 t \right] F(k_\perp T, k_\parallel t) . \quad (2)$$

Here $k^2 = k_\parallel^2 + k_\perp^2$: ($k_{\parallel, \perp}$: parallel and normal to the magnetic field respectively), ω_p is the plasma frequency, Ω is the cyclotron frequency (for electrons $\Omega_e < 0$). The function F is the Fourier transform in $v_\parallel$, with argument $k_\parallel t$ and 2π times the Fourier-Bessel transform of zero order in $v_\perp$, with argument $k_\perp T$ of the unperturbed distribution function f_0 :

$$F(k_\perp T, k_\parallel t) = 2\pi \int_0^\infty dv_\perp v_\perp J_0(k_\perp T v_\perp) \int_{-\infty}^\infty dv_\parallel \exp(-ik_\parallel v_\parallel t) f_0(v_\perp, v_\parallel) , \quad (3)$$

where T is a periodic function of t , $T(t) = |\sin(\Omega t/2)| / (\Omega/2)$. When $k_\parallel = 0$, note that H is a periodic function of time. This points towards the absence of Landau damping for

perpendicular waves (Bernstein modes). When $k_{\parallel} \neq 0$, the response shows a time decay inversely correlated with the velocity spread in $v_{\parallel}$. The term

$$\begin{aligned} g(t) = & \left(\frac{4\pi e}{k^2} \right) \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} dv_{\perp} v_{\perp} \times \\ & \times \int_0^{2\pi} d\alpha \exp \left\{ \left(\frac{-ik_{\perp} v_{\perp}}{\Omega} \right) [\sin(\Omega t + \alpha) - \sin \alpha] - ik_{\parallel} v_{\parallel} t \right\} \times \\ & \times f_1(\underline{k}, v_x = v_{\perp} \cos(\Omega t + \alpha), v_y = v_{\perp} \sin(\Omega t + \alpha), v_{\parallel}, t = 0) \end{aligned} \quad (4)$$

is a datum which depends on the initial value of the perturbation of the distribution function, f_1 , at $t = 0$. Similar terms appear on the r.h.s. of the following equations.

2. Electromagnetic modes of low frequency. This range includes the magnetohydrodynamic waves (Alfvén waves, whistlers, etc.) up to the cyclotronic resonances. Electromagnetic modes propagating along the magnetic field for frequencies lower than the electron plasma frequency, $k_{\perp} = 0$, are circularly polarized $E_{\pm}(t) = E_x(t) \pm iE_y(t)$. The corresponding integral equation is

$$(c^2 k^2 + \omega_{pe}^2 + \omega_{pi}^2) E_{\pm} + 4\pi \int_0^t d\tau \left(\frac{\partial \sigma_{e\pm}}{\partial \tau} + \frac{\partial \sigma_{i\pm}}{\partial \tau} \right) E_{\pm}(t - \tau) = g_{\pm}(t), \quad (5)$$

with

$$\begin{aligned} 4\pi \left(\frac{\partial}{\partial t} \right) \sigma_{\pm} &= \omega_p^2 \left\{ \frac{\partial}{\partial t} [\exp(\mp i\Omega t) G(kt)] - \exp(\mp i\Omega t) \left[\frac{\partial}{\partial t} G(kt) + k^2 t L(kt) \right] \right\}, \quad (6) \\ G &= \mathcal{F}[2\pi \int_0^{\infty} dv_{\perp} v_{\perp} f_0], \\ L &= \frac{1}{2} \mathcal{F}[2\pi \int_0^{\infty} dv_{\perp} v_{\perp}^2 f_0], \end{aligned}$$

where $\mathcal{F}$ denotes Fourier transform. When f_0 is isotropic, i.e. $f_0 = f_0(|\underline{v}|)$, the second term in the r.h.s. of Eq. (6) is zero.

4. General theoretical results

4.1 Convolution algebra

For the theory of Volterra integral equations see refs. (Tricomi, 1957; Brunner, van der Houwen, 1986). The operational calculus used here is based on Erdélyi's monograph (Erdélyi, 1966). The convolution product is defined by

$$H * \phi = \int_0^t H(t) \phi(t - \tau) d\tau, \quad t \geq 0 \quad (7)$$

Powers of H denote convolutions of H with itself, like $H^2 = H * H$, $H^n = H^{n-1} * H$. The symbol h stands for the constant function of value 1, and acts like an integral operator in $h * \phi$. We have $h^2 = t$, $h^{n+1} = t^n/n!$. A function is said to be *locally integrable* (LI) when it is absolutely integrable over a finite interval $(0, t)$, for all t . Given two continuous functions

$a(t)$, $b(t)$ the equation $a * z = b$ defines a unique convolution quotient $z = b/a$. The set of continuous functions with the convolution product forms a commutative ring without identity element. The ring can be extended to an algebraic field with the introduction of the convolution quotient. The convolution quotient b/a is represented by any ordered pair of continuous functions $b/a = (a, b) = (c, d)$ satisfying $a * d = b * c$. Convolution quotients may define generalized functions. The identity for the convolution product or delta function, is given by $\mathbf{1} = (a, a) = (b, b) = \dots$ with $a, b, \dots$ continuous. In fact, $a * \mathbf{1} = a$ corresponds to $a * b/b = a$, since $a * b = a * b$. The quotient $h/h^2 = s$ is the differential operator; note that $\mathbf{1} = s * h$, and $s * a = a' + a(0)$, if a possesses a LI derivative. An important formula for our applications is $(s - \alpha)^{-1} = \exp(\alpha t)$, which follows from $(s - \alpha) * \exp(\alpha t) = \mathbf{1}$, where α is a constant. This can be extended to $(s - \alpha)^{-n} = t^{n-1} \exp(\alpha t)/(n-1)!$.

4.2 Resolvent

The Volterra equation for the electric potential can be written in operational form as

$$(\mathbf{1} + H) * \phi = g . \quad (8)$$

The solution is defined by the convolution quotient

$$\phi = g/(\mathbf{1} + H) . \quad (9)$$

We define the resolvent kernel R as

$$R = \mathbf{1}/(\mathbf{1} + H) - \mathbf{1} . \quad (10)$$

It is easy to verify, by performing convolution products, that

$$R = \sum_{m=1}^{\infty} (-1)^m H^m . \quad (11)$$

Hence, we may write

$$\phi(t) = (\mathbf{1} + R) * g . \quad (12)$$

It can be proved (Erdélyi, 1966) that: i) if H is LI then $H^n \rightarrow 0$ for $n \rightarrow \infty$; ii) if H is LI the resolvent converges absolutely and uniformly on a finite time interval, and R is LI; iii) When H is a continuous function then R is a continuous function. The solution ϕ exists and is unique for any given $g \in \text{LI}$. Note that due to the physical origin of H and g , i.e. they are Fourier transforms in velocity space of distribution functions, then H and ϕ are continuous functions.

4.3 Examples

For a cold plasma $f_0(v) = \delta(v)$. Taking $k_{\perp} = 0$ we have for electrostatic modes,

$$H(t) = \omega_p^2 t \quad , \quad R = -\omega_p \sin(\omega_p t) . \quad (13)$$

For a Cauchy–Lorentz function $f_0(v) = a/(a^2 + v^2)$, and $k_{\perp} = 0$ we have

$$H(t) = \omega_p^2 t \exp(-akt) \quad , \quad R(t) = -\omega_p \sin(\omega_p t) \exp(-akt) . \quad (14)$$

For a cold plasma in a magnetic field (with $k_{\perp} \neq 0$) we have $F(k_{\perp}T, k_{\parallel}t) = 1$ and

$$H(t) = \left(\frac{\omega_p}{k}\right)^2 \left[k_{\perp}^2 \left(\frac{\sin \Omega t}{\Omega} \right) + k_{\parallel}^2 t \right] .$$

The resolvent is given by

$$R(t) = -\frac{\omega_p k_{\parallel}}{k} \sin \left(\frac{\omega_p k_{\parallel} t}{k} \right) - \left(\frac{\omega_p k_{\perp}}{k} \right)^2 \frac{\sin \nu t}{\nu} , \quad (15)$$

with

$$\nu = \sqrt{\left(\frac{\omega_p k_{\perp}}{k} \right)^2 + \Omega^2} .$$

If the plasma is not cold in the parallel velocity $v_{\parallel}$, and has a Cauchy–Lorentz distribution for this component, $f_0 = (1/2\pi v_{\perp})\delta(v_{\perp})a/(a^2 + v_{\parallel}^2)$, then $F = \exp(-ak_{\parallel}t)$ so that

$$H(t) = H_c \exp(-ak_{\parallel}t) , \quad R = R_c \exp(-ak_{\parallel}t) ,$$

where the subindex c indicates the previous cold case formulas.

4.4 Meaning and properties of resolvents

Replacing $g = -H$ in (12) we immediately find $\phi = R$, so that we can state

$$(\mathbf{1} + H) * R = -H \quad (16)$$

(Fredholm–Volterra identity). This equation gives the physical meaning of the resolvent kernel: it is the electric potential which arises in the plasma when the initial perturbation is $g = -H$. This happens when $f_1(\underline{k}, \underline{v}, 0)$ is proportional to $\partial f_0 / \partial \underline{v}$.

It can be shown that in an homogeneous plasma the linear perturbation $f_1(\underline{x}, \underline{v}, t)$ can be written in the form $f_1 = -\underline{v}_1 \cdot \partial f_0 / \partial \underline{v}$, for all t . Here $\underline{v}_1(\underline{x}, \underline{v}, t)$ is the velocity perturbation of particle orbits at the position $(\underline{x}, \underline{v})$ in phase space. This expression holds under the condition that the initial displacement of the orbits $\underline{v}_1(\underline{x}_0, \underline{v}_0, 0)$ is continuous (Nardi et al., 1987a,b). That is, sources or sinks of particles, like the introduction of beams at $t = 0$, are excluded. We may call a perturbation of this kind “normal” or “natural”. This implies that, far from being arbitrary, $f_1(\underline{x}, \underline{v}, 0)$ in this case is closely related to the distribution function $f_0(\underline{v})$. Therefore we see that the resolvent potential $\phi = R$ corresponds to a normal perturbation with $\underline{v}_1(\underline{x}_0, \underline{v}_0, 0)$ independent of $\underline{v}_0$.

Let us denote by U the solution $\phi = U$ corresponding to a constant unit datum $g = 1$, i.e.

$$(\mathbf{1} + H) * U = h . \quad (17)$$

Now consider any problem with a differentiable datum g

$$(\mathbf{1} + H) * \phi = g .$$

The potential is given then by

$$\phi = U * s * g = g(0)U + U * g' . \quad (18)$$

so that U also acts like a kind of resolvent. In fact, $U = h/(\mathbf{1} + H)$, $\phi = \mathbf{1}/(\mathbf{1} + H) * g = h/(\mathbf{1} + H) * s * g$. We immediately see, multiplying by s , that $s * U = \mathbf{1} + R$, i.e. $R = \partial U / \partial t$.

4.5 Growth bounds for the potential

This operator algebra allows us to find bounds for the growth of $\phi(t)$. We give an example here. Assume that $|H(t)| \leq \gamma^2 t$ and $|g(t)| \leq C$, where γ^2 and C are constants, which holds for the plasma wave problem. Define

$$a(t) = (\mathbf{1} - \gamma^2 t) * |\phi|.$$

From $(\mathbf{1} + H) * \phi = g$, it follows that $(\mathbf{1} - \gamma^2 t) * |\phi| \leq C$. Solving for ϕ in terms of $a(t)$, with a resolvent, we have

$$|\phi| = (\mathbf{1} + R) * a.$$

It is easy to verify, performing convolution products, that the resolvent of the kernel $-\gamma^2 t$, is $R = \gamma \sinh \gamma t$, i.e. that $(\mathbf{1} + \gamma \sinh \gamma t) * (\mathbf{1} - \gamma^2 t) = \mathbf{1}$. Hence, $|\phi| \leq C(\mathbf{1} + R) * h = C \cosh \gamma t$. This gives an upper bound for the solution for all t . We shall see that $\gamma = \omega_p$ for quasi-electrostatic waves in a magnetized plasma, thus we conclude that no instability can have a growth rate larger than ω_p for this class of modes.

4.6 Plasma stability

The bound derived above also holds for stable solutions. In the following section we shall see that the asymptotic behaviour of $\phi(t)$ for $t \rightarrow \infty$ can be described for stable plasmas. To determine whether $f_0(\underline{v})$ corresponds to a stable plasma we must exclude the existence of zeros of the dielectric function $\varepsilon(k, p)$ with $\text{Re } p > 0$. This can be investigated as follows. Let $\overline{H}(p) = \int_0^\infty \exp(-pt) H(t) dt$ denote the Laplace transform of $H(t)$, then $\varepsilon(k, p) = 1 + \overline{H}(p)$. The existence of simple roots of $1 + \overline{H}(p)$ at points $p_0, p_1, \dots$ ($\text{Re } p_0 > \text{Re } p_1 > \dots$) may be studied noting that the root with the largest real part may be expressed as $p_0 = I$, with $I(\alpha)$ defined by

$$I(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dy \frac{\alpha + iy}{1 + \overline{H}(\alpha + iy)} \frac{d}{dy} \overline{H}(\alpha + iy), \quad (19)$$

when α is such that $\text{Re } p_0 > \alpha > \text{Re } p_1$ (Baldwin et al., 1971). Thus a numerical procedure may be applied. It suffices to compute $I(\alpha)$ for $\alpha = \gamma$, the growth bound established by our previous theorem, and $\alpha = 0$. The existence of a root with $\text{Re } p_0 \geq 0$ will produce a change in the value of I , $I(\gamma) \neq I(0)$. When $I(\gamma) = I(0)$ it follows that $1 + \overline{H}(p) \neq 0$ in the half plane $\text{Re } p \geq 0$, including the contour $\text{Re } p = 0$. A plasma is said to be linearly stable when $f_0(v)$ is such that $1 + \overline{H}(p)$ has no zeros with $\text{Re } p > 0$, i.e. no perturbation $\phi(t)$ grows without bounds as $t \rightarrow \infty$.

4.7 Asymptotic values of the solution for stable plasmas

A celebrated theorem by Paley and Wiener (1934) (Brunner and van der Houwen, 1986) may be used to predict the asymptotic value of ϕ . Let $\int_0^\infty |H(t)| dt < \infty$, i.e. $H \in L^1(R_+)$, then it can be proved that the resolvent R is also $L^1(R_+)$, if and only if, $1 + \overline{H}(p)$ has no

roots in the right half plane $\text{Re } p \geq 0$ (including the contour $\text{Re } p = 0$). That is, when H is absolutely integrable for all t , then the resolvent for stable plasmas is also absolutely integrable for $t \in (0, \infty)$. The main consequence of this property is the following: when $g(\infty) = \lim_{t \rightarrow \infty} g(t)$ exists, then it can be shown that

$$\phi(\infty) = \frac{g(\infty)}{1 + \int_0^\infty H(t) dt} . \quad (20)$$

We may note that the question of the conditions for the Landau damping of plasma waves, which has been debated in the literature (Backus, 1960; Montgomery, 1971; Hayes, 1961; 1963; Pécseli, 1985), can be stated in a neat form with this approach. When the plasma is stable, square integrable perturbations $f_1 \in L^2$ produce square integrable $g(t)$. Thus, for these perturbations, $g(\infty) = 0$ (Riemann–Lebesgue lemma), so that $\phi(\infty) = 0$, i.e. these perturbations are Landau damped. Normal excitations (which do not include beams, see Section 4.4) are always damped, provided $\partial f_0 / \partial v \in L^2$. However, more general initial perturbations f_1 with sharp particle density (delta function: $\delta(v)$), or plasmas with discontinuities in f_0 may give $g(\infty) \neq 0$, and therefore $E(\infty) \neq 0$.

4.8 Arbitrary evolution of $\phi(t)$ in stable plasmas

The time asymptotic representation of the solution is not always that predicted by Landau, and this fact depends on the function $g(t)$ as we have seen. In the references (Hayes, 1961; 1963; Pécseli, 1985; Weitzner, 1962) it is shown for electrostatic modes without magnetic field, that the time decay of $\phi(t)$ may in principle take any value, a result which may seem surprising at first sight. It requires a rather elaborate procedure to put this fact in evidence working with the Van Kampen–Case method. We want to note here that this arbitrariness seems natural in the perspective of Volterra equations. It is simply a consequence of the unique relation between $\phi(t)$ and $g(t)$, via the integral equation. We may choose any bounded function $\phi(t)$ for the potential in a stable plasma, and determine the datum $g(t)$ from $g = (\mathbf{1} + H) * \phi$. We obtain then the corresponding initial perturbation $f_1(k, v, 0)$ by an inverse Fourier transform. Consider, as an example, electronic waves in a Maxwellian plasma (without magnetic field), and suppose we want to obtain a constant electric field, $E(t) = \mathcal{E} = \text{constant}$. Here we have $f_0 = (1/\sqrt{2\pi}v_t) \exp[-(1/2)(v/v_t)^2]$, $H = \omega_p^2 t \exp[-(1/2)(kv_t t)^2]$. We find

$$g(t) = \mathcal{E} \left\{ 1 + \left(\frac{\omega_p}{kv_t} \right)^2 (1 - \exp[-(kv_t t)^2]) \right\} ,$$

and the inverse Fourier transform leads to

$$f_1(k, v, 0) = C_1 \delta(v) + C_2 \frac{1}{\sqrt{2\pi}v_t} \exp \left[- \left(\frac{v}{v_t} \right)^2 \right] , \quad (21)$$

with

$$\begin{aligned} C_1 &= -\frac{ik\mathcal{E}}{4\pi e} \left[1 + \left(\frac{\omega_p}{kv_t} \right)^2 \right] , \\ C_2 &= \frac{ik\mathcal{E}}{4\pi e} \left(\frac{\omega_p}{kv_t} \right)^2 . \end{aligned}$$

Given this (singular) initial perturbation the electric field in the plasma stays constant in time.

5. Bounds for low frequency plasma modes

We present here results on growth bounds for plasma modes in a magnetic field governed by the integral equations presented in Section 3. By procedures similar to those outlined in Section 4.5 the following theorems hold for solutions of Eq. (1) (Gnavi and Gratton, 1987b).

1. Let $|H(t)| \leq \gamma^2 t$ and $|g(t)| \leq B$, where γ and B are constants. Then, $\phi(t) \leq B \cosh \gamma t$ for all finite t .
2. If $|H(t)| \leq \alpha$ and $|g(t)| \leq B$, then $|\phi(t)| \leq B \exp(\alpha t)$ for all finite t (α , B constants).
3. If $|H(t)| \leq 2\alpha + \beta^2 t$ and $|g(t)| \leq B$, then $\phi(t) \leq B \exp(\alpha t) [\cosh(\mu t) + (\alpha/\mu) \sinh(\mu t)]$, where $\mu = \sqrt{(\alpha^2 + \beta^2)}$, for all finite t .

a) Consider first electron motion only. Since $|F| \leq 1$ always, it follows that $|H| \leq \omega_p^2 t |F(k_\perp T, k_\parallel t)| \leq \omega_p^2 t$, independently of f_0 . Hence, no electronic quasi-electrostatic instability may grow faster than $\exp(\omega_p t)$. This extends the validity of a result due to Backus (1960) to a magnetized plasma with arbitrary $\underline{k}$.

b) When $k_\parallel = 0$, however, we also have $|H| \leq \omega_p^2 / |\Omega|$. We obtain then $\phi \leq C \exp(\omega_p t / |\Omega|)$, which is a better bound when $|\omega_p / \Omega| < 1$. For plasma regimes in which $|\omega_p / \Omega| \ll 1$, the growth rate for modes perpendicular to $\underline{B}_0$ is considerably reduced compared to that of parallel modes. These results cover the range of the electron plasma waves, upper hybrid and electron Bernstein modes.

c) We may consider the ion-acoustic frequencies including the ion response. It is often possible to take the electrons in a Boltzmann equilibrium with the wave potential. If $k_\parallel \neq 0$, the contribution of the electrons may be described by a term $\phi / (k \lambda_D)^2$ (λ_D : Debye wavelength) and we obtain Eq. (1) with $H = H_i / (1 + k^2 \lambda_D^2)$ where H_i , the ion kernel, is given by Eq. (2). The contribution of $g(t)$ is divided by the same factor as H_i . The upper growth bound for the solutions is $\gamma = kc_s / \sqrt{1 + k^2 \lambda_D^2}$, where $c_s = \sqrt{\theta_e / m_i}$ is the ion sound speed. This extends to magnetized plasmas a growth bound due to Pécseli (Pécseli, 1975), who observed that for long wavelengths $\gamma \rightarrow 0$.

d) When $k_\parallel = 0$ (lower hybrid limit) the electron contribution is modified and a term $\phi(\omega_{pe} / \Omega_e)^2$ must be added to Eq. (1) instead. In this case, we obtain $\gamma = \omega_{pi} / \sqrt{1 + (\omega_{pe} / \Omega_e)^2}$, or when $\omega_{pi} / \Omega_i < 1$, $\gamma = \omega_{pi}^2 / \Omega_i \sqrt{1 + (\omega_{pe} / \Omega_e)^2}$, so that the growth rates are reduced by strong fields.

e) For low frequency electromagnetic modes we have for isotropic f_0 , $4\pi |\partial \sigma_\pm / \partial t| \leq \omega_p^2 \omega_{1\pm}$, where $\omega_{1\pm} = |k_\parallel| \int_{-\infty}^{\infty} dv_\parallel |v_\parallel| \pm \Omega / k_\parallel |2\pi \int_0^\infty dv_\perp v_\perp f_0$. These expressions apply to electrons and ions. Using theorem 2 in Eq. (5), the growth rate limit is $\gamma = \Gamma_e \omega_{1\pm e} + \Gamma_i \omega_{1\pm i}$, with $\Gamma_{e,i} = \omega_{pe,i}^2 / (\omega_{pe}^2 + \omega_{pi}^2 + c^2 k_\parallel^2)$. The bound decreases as $k_\parallel$ increases. For nonisotropic f_0 we get $4\pi |\partial \sigma_\pm / \partial t| \leq |\Omega| + \omega_2^2 t$, where $\omega_2^2 = (1/2) k_\parallel^2 \int d^3 v v_\perp^2 f_0$. These expressions hold for ions and electrons. Applying theorem 3 to Eq. (5) we obtain the growth rate bound

$\gamma = \alpha + \sqrt{\alpha^2 + \beta^2}$, with $\alpha = (\Gamma_e |\Omega_e| + \Gamma_i \Omega_i)/2$, $\beta^2 = \Gamma_e \omega_{2e}^2 + \Gamma_i \omega_{2i}^2$. The limits of growth given here hold for any distribution function and are valid for all times, during the linear phase (including the early stages of a perturbation, both for stable and unstable modes). Some bounds become very stringent for particular sets of parameters. By adding specific information about the distribution function, the limits may be improved.

6. Numerical solutions

6.1 Electron plasma waves

Numerical solutions are easily obtained using available codes for Volterra equations of the second kind. As a preliminary example, we report the electric field for electron plasma waves in a Maxwellian plasma ($k_\perp = 0$), $f_0(v_\parallel) = (1/\sqrt{2\pi}v_t) \exp[-(1/2)(v_\parallel/v_t)^2]$, with a polarization response $H(t) = \omega_p^2 t \exp[-(1/2)(kv_t t)^2]$. We take $g = A/k$, where A is a constant. The solution ϕ is related to the resolvent U (see 4.4). Hence by storing this numerical solution, we may subsequently compute the electric field for any arbitrary g . Figures 1 to 3 show the numerical results. The time axis is $\tau = \omega_p t$, so that a plasma period is 2π . The ordinate is $E(t)/E(0)$. The wavenumber is given in units of $k_D = 1/\lambda_D = \omega_p/v_t$, the inverse of the Debye length. For $k/k_D = \sqrt{20}$ we have a strong damping and no oscillations. At $k/k_D = \sqrt{2}$ an incipient oscillation is observed, but the damping is still very large. For $k/k_D = \sqrt{0.2}$ we may see several plasma oscillations which decay in time. Note that in this case $g(\infty) \neq 0$ and therefore $E(\infty) \neq 0$, in accordance with the Paley–Wiener result, Section 4.7.

6.2 Electron Bernstein modes in a magnetospheric plasma

For electron Bernstein modes in a magnetospheric plasma, the evolution of the potential may be described in terms of the electron response function only. In the case of spatial anisotropy, the unperturbed distribution function f_0 may be of Maxwellian type

$$f_0 = \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{c_\parallel c_\perp^2}\right) \exp \left\{ -\left(\frac{1}{2}\right) \left[\left(\frac{v_\perp}{c_\perp}\right)^2 + \left(\frac{v_\parallel}{c_\parallel}\right)^2 \right] \right\}, \quad (22)$$

with $c_\perp^2 = T_\perp/m$ and $c_\parallel^2 = T_\parallel/m$, and the Fourier–Bessel transform is

$$F(k_\perp T, k_\parallel t) = \exp \left\{ -\left(\frac{1}{2}\right) [(k_\perp c_\perp T)^2 + (k_\parallel c_\parallel t)^2] \right\}. \quad (23)$$

For modes propagating perpendicular to the magnetic field ($k_\parallel = 0$) this reduces to

$$F(k_\perp T, 0) = \exp \left\{ -\frac{1}{2} \left[\frac{2k_\perp c_\perp}{\Omega} \sin \left(\frac{\Omega t}{2} \right) \right]^2 \right\},$$

so that one obtains, with time measured in units of $1/|\Omega_e|$

$$\phi(t) + \left(\frac{\omega_{pe}}{\Omega_e}\right)^2 \int_0^t d\xi \sin \xi \exp \left\{ -\frac{1}{2} \left[2k R_{Le} \sin \left(\frac{\xi}{2} \right) \right]^2 \right\} \phi(t - \xi) = g(t). \quad (24)$$

For the perturbation distribution function f_1 we choose:

$$f_1(\underline{k}, v_{\parallel}, v_{\perp}, t = 0) = \alpha \left(\frac{1}{2\pi v_{\perp}} \right) \delta(v_{\parallel}) \delta(v_{\perp}) ,$$

leading to a constant value $g(t) = \alpha 4\pi e/k^2$; and

$$f_1(\underline{k}, v_{\parallel}, v_{\perp}, t = 0) = \alpha \left(\frac{1}{4a_{\parallel}a_{\perp}} \right) \delta(v_y) \exp \left(-\frac{|v_{\parallel}|}{a_{\parallel}} \right) \exp \left(-\frac{|v_x|}{a_{\perp}} \right) ,$$

which gives

$$g(t) = \alpha \left(\frac{4\pi e}{k^2} \right) \left[1 + \left(\frac{k_{\perp} a_{\perp}}{|\Omega_e|} \sin t \right)^2 \right]^{-1} ,$$

where we have set $k_{\parallel} = 0$. When $k \ll 1/R_{Le}$ the resolvent of this integral equation is given in Section 4.3. For constant g the solution is

$$\phi(t) = \frac{a}{1 + \mu} [1 + \mu \cos(\sqrt{1 + \mu} t)]$$

with $\mu = (\omega_p/\Omega)^2$. We see that the oscillatory behaviour as well as the sign of $\phi(t)$ depend strongly on μ .

Let us consider typical parameter values for magnetospheric conditions just inside the plasmopause boundary. We take $n_e \approx 100 \text{ cm}^{-3}$ for the density, $B \approx 250\gamma$ for the magnetic field, and $E_e \approx 0.2eV$ for the thermal electron energy. It follows then that $\omega_{pe} \approx 6 \cdot 10^5 \text{ rad/sec}$, $|\Omega_e| \approx 4 \cdot 10^4 \text{ rad/sec}$ and $c_{\perp e} \approx 2 \cdot 10^7 \text{ cm/sec}$, so that $(\omega_{pe}/\Omega_e)^2 \approx 200$ and the electron Larmor radius is given by $R_{Le} = c_{\perp e}/|\Omega_e| \approx 500 \text{ cm}$. Figures 4 and 5 show $\phi(t)/\phi(0)$ plotted versus time with $g(t) = 0.1/k^2$, and $g(t) = (0.1/k^2)[1 + 2.5 \times 10^5 k^2 \sin^2 t]^{-1}$, respectively, where we have set $a_{\perp} = c_{\perp e}$. In both cases we have used $k = 8 \cdot 10^{-3} \text{ cm}^{-1}$. With this choice $\lambda = \pi R_{Le}/2$, and the exponential function in the kernel varies from 1 to $\exp(-32)$ during the cyclotron period. We observe that the oscillations show the upper hybrid frequency $\omega = \sqrt{1 + \mu}|\Omega_e|$, while the amplitude is modulated over a cyclotron period.

We plan to present more extensive numerical studies of the initial value problem of plasma waves in a separate paper. Note that the Volterra equations approach can also be extended to finite amplitude waves by using the proper nonlinear response kernels.

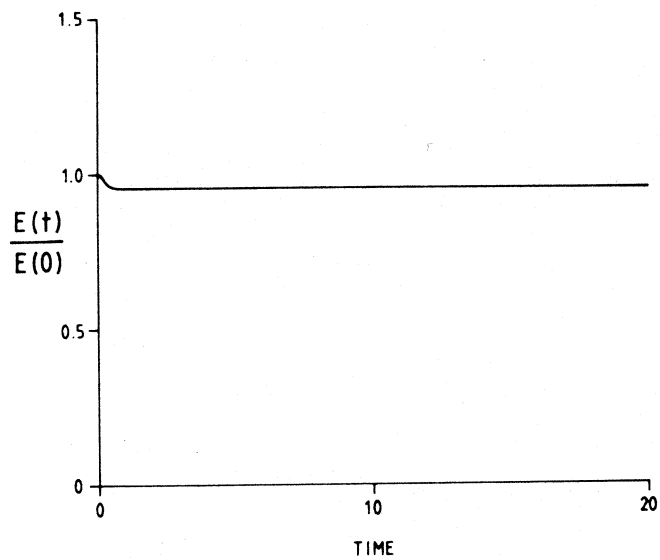


Fig. 1: Electric field $E(t)$ for electron plasma waves in a Maxwellian plasma as a function of time. The electric field is normalized to its value at $t = 0$, and the time is scaled in multiples of $1/\omega_p$, where ω_p is the plasma frequency. We used $k/k_D = \sqrt{20}$.

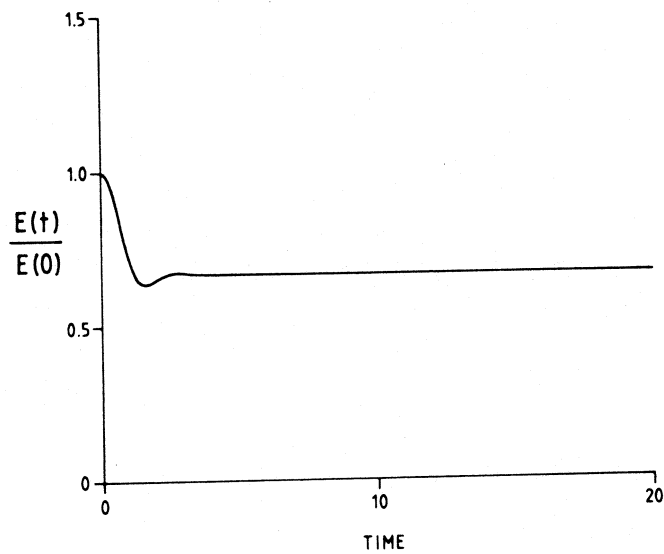


Fig. 2: As for Figure 1, with $k/k_D = \sqrt{2}$.

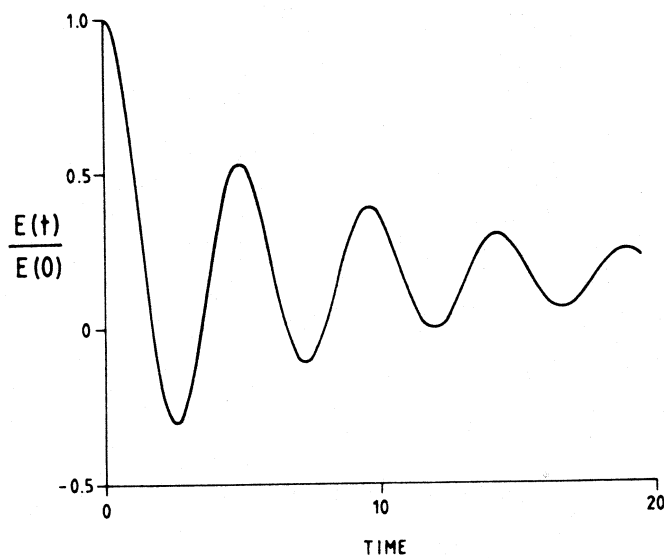


Fig. 3: As for Figure 1, with $k/k_D = \sqrt{0.2}$.

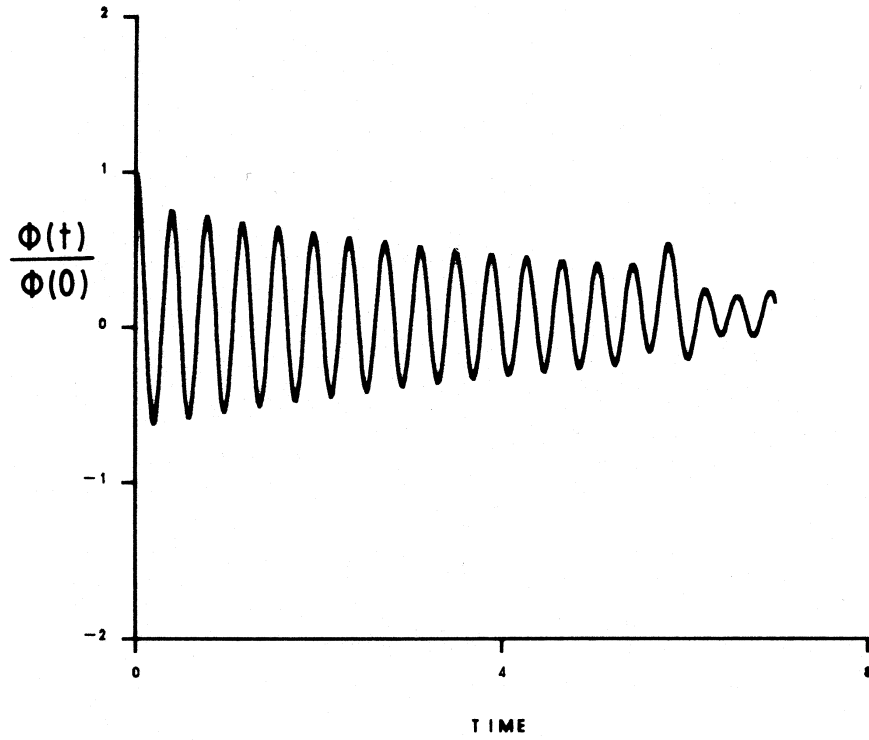


Fig. 4: Electric potential $\phi(t)$ for magnetospheric Bernstein modes as a function of time. The potential is normalized to its value at $t = 0$, and the time is scaled in multiples of $1/|\Omega_e|$, where Ω_e is the electron cyclotron frequency. We used $g(t) = 0.1/k^2$ and $k = 8 \cdot 10^{-3} \text{ cm}^{-1}$.

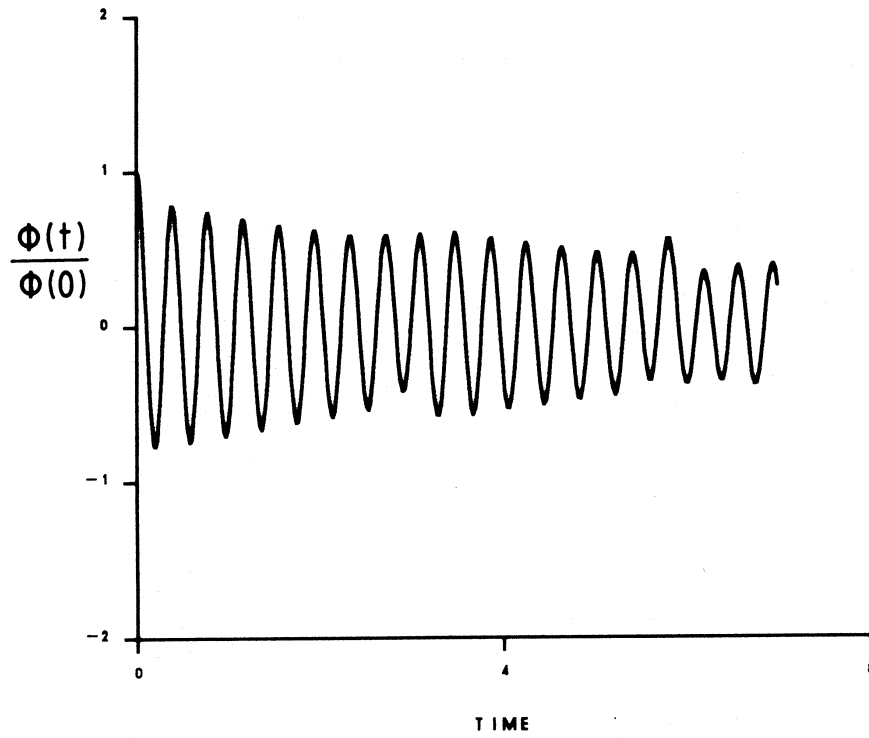


Fig. 5: As for Figure 4, except that $g(t) = (0.1/k^2)[1 + 2.5 \times 10^5 k^2 \sin^2 t]^{-1}$.

