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Generalizations of Implication Algebras

By

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Abstract

Implication algebras, originally introduced in order to study algebraic properties of the implication operation in Boolean algebras, are generalized and it is shown that these more general algebras are in one-to-one correspondence to semilattices with 1 the principal filters of which are posets with an antitone involution, respectively to commutative directoids with 1 the principal filters of which are posets with a switching involution.

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In order to study algebraic properties of the implication operation in Boolean algebras J. C. ABBOTT introduced the notion of an implication algebra (cf. [1]). He showed that these algebras are in one-to-one correspondence to join-semilattices with 1 the principal filters of which are Boolean algebras. These algebras were generalized, e.g. in [2], [8] and [9], where corresponding results were achieved.

Let us mention that also other types of implication in non-classical logic were treated in the literature (see e.g. [4]–[7]). However, these

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can be unified by a more generalized approach which will be presented here.

The aim of this paper is to further generalize the concept of an implication algebra to algebras satisfying weaker conditions. It turns out that by the constructions originally used by J. C. Abbott these more general algebras are in one-to-one correspondence to semilattices with 1 the principal filters of which are posets with an antitone involution, respectively to directoids with 1 the principal filters of which are posets with a switching involution.

1. Implication Algebras, Orthoimplication Algebras and Orthomodular Implication Algebras

We start our investigations by repeating the definition of an implication algebra and its connection to certain join-semilattices having an additional structure.

Definition 1.1. Let $\mathcal{A} = (A, \cdot, 1)$ be an algebra of type $(2, 0)$. $\mathcal{A}$ is called an *implication algebra* (cf. [1]) if it satisfies

$$xx = 1,$$
$$(xy)x = x,$$
$$(xy)y = (yx)x$$

and

$$x(yz) = y(xz).$$

Remark 1.2. The nullary operation 1 can be dropped from the family of fundamental operations of an implication algebra since due to the first identity in the definition it is an algebraic constant.

The following theorem was proved in [1]:

Theorem 1.3. *Let $\mathcal{A} = (A, \cdot, 1)$ be an implication algebra. Define*

$$x \vee y := (xy)y \qquad \text{and} \qquad x^y := xy$$

for all $x, y \in A$. Then $\mathbf{S}(\mathcal{A}) := (A, \vee, (^x; x \in A), 1)$ is an algebra such that $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in A$ $([x, 1], \leq, ^x)$ is a Boolean algebra. Conversely, let $\mathcal{S} := (S, \vee, (^x; x \in S), 1)$ be an algebra such that $(S, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in S$ $([x, 1], \leq, ^x)$ is a Boolean algebra. Define

$$xy := (x \vee y)^y$$

for all $x, y \in S$. *Then* $\mathbf{A}(\mathcal{S}) := (S, \cdot, 1)$ *is an implication algebra. Moreover,* $\mathbf{A}(\mathbf{S}(\mathcal{A})) = \mathcal{A}$ *and* $\mathbf{S}(\mathbf{A}(\mathcal{S})) = \mathcal{S}$ *for every implication algebra* $\mathcal{A}$ *and every algebra* $\mathcal{S} = (S, \vee, ({}^x; x \in S), 1)$ *such that* $(S, \vee, 1)$ *is a join-semilattice with greatest element* 1 *and for every* $x \in S$ $([x, 1], \leq, {}^x)$ *is a Boolean algebra.*

The notion of an implication algebra was generalized to the notion of an orthoimplication algebra, respectively orthomodular implication algebra, as follows:

Definition 1.4. Let $\mathcal{A} = (A, \cdot, 1)$ be an algebra of type $(2, 0)$. $\mathcal{A}$ is called an *orthoimplication algebra* (cf. [2]) if it satisfies

$$\begin{aligned} xx &= 1, \\ (xy)x &= x, \\ (xy)y &= (yx)x \end{aligned}$$

and

$$x((yx)z) = xz.$$

Definition 1.5. Let $\mathcal{A} = (A, \cdot, 1)$ be an algebra of type $(2, 0)$. $\mathcal{A}$ is called an *orthomodular implication algebra* (cf. [8] and [9]) if it satisfies

$$\begin{aligned} xx &= 1, \\ (xy)x &= x, \\ (xy)y &= (yx)x, \\ (((xy)y)z)(xz) &= 1 \end{aligned}$$

and

$$((((((xy)y)z)x)x)z)x)x = (((xy)y)z)z.$$

In [2], [8] and [9] it was proved that orthoimplication algebras, respectively orthomodular implication algebras, correspond to join-semilattices with 1 the principal filters of which are orthomodular lattices satisfying the compatibility condition ($x \leq y \leq z$ implies $z^y = z^x \vee y$) respectively to join-semilattices with 1 the principal filters of which are orthomodular lattices. These correspondences are one-to-one and completely analogous to that proved by J. C. Abbott in [1] for implication algebras.

2. *I*-Algebras

We now define a new more general type of implication algebras:

Definition 2.1. Let $\mathcal{A} = (A, \cdot, 1)$ be an algebra of type $(2, 0)$. $\mathcal{A}$ is called a *strong I-algebra* if it satisfies

$$1x = x, \tag{S1}$$

$$xx = 1, \tag{S2}$$

$$x(yx) = 1, \tag{S3}$$

$$(xy)y = (yx)x \tag{S4}$$

and

$$(((xy)y)z)(xz) = 1. \tag{S5}$$

Remark 2.2. The class of all strong I-algebras forms a variety.

For the following theorem we need the definition of an antitone involution of a poset.

Definition 2.3. Let $(P, \leq)$ be a poset and $f \colon P \to P$. f is called an *antitone involution* of $(P, \leq)$ if $f(x) \geq f(y)$ whenever both $x, y \in P$ and $x \leq y$ and if $f(f(x)) = x$ for all $x \in P$.

Now the following result can be proved:

Theorem 2.4. *Let $\mathcal{A} = (A, \cdot, 1)$ be a strong I-algebra. Define*

$$x \vee y := (xy)y \qquad \text{and} \qquad x^y := xy$$

for all $x, y \in A$. Then $\mathbf{S}(\mathcal{A}) := (A, \vee, (^x; x \in A), 1)$ is an algebra such that $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in A$ $([x, 1], \leq, ^x)$ is a poset with an antitone involution where $\leq$ denotes the partial order induced by $\vee$. Conversely, let $\mathcal{S} := (S, \vee, (^x; x \in S), 1)$ be an algebra such that $(S, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in S$ $([x, 1], \leq, ^x)$ is a poset with an antitone involution. Define

$$xy := (x \vee y)^y$$

for all $x, y \in S$. Then $\mathbf{A}(\mathcal{S}) := (S, \cdot, 1)$ is a strong I-algebra. Moreover, $\mathbf{A}(\mathbf{S}(\mathcal{A})) = \mathcal{A}$ and $\mathbf{S}(\mathbf{A}(\mathcal{S})) = \mathcal{S}$ for every strong I-algebra $\mathcal{A}$ and every algebra $\mathcal{S} = (S, \vee, (^x; x \in S), 1)$ such that $(S, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in S$ $([x, 1], \leq, ^x)$ is a poset with an antitone involution.

Proof. Assume $(A, \cdot, 1)$ to be a strong I-algebra and for all $x, y \in A$ define $x \leq y$ if $xy = 1$, $x \vee y := (xy)y$ and $x^y := xy$. Because of (S2), $\leq$ is reflexive. If $a \leq b$ and $b \leq a$ then

$$a = 1a = (ba)a = (ab)b = 1b = b$$

according to (S1) and (S4) proving antisymmetry of $\leq$. If $a \leq b \leq c$ then

$$ac = 1(ac) = (bc)(ac) = ((1b)c)(ac) = (((ab)b)c)(ac) = 1$$

according to (S1) and (S5), i.e., $a \leq c$. This shows transitivity of $\leq$. Hence $(A, \leq)$ is a poset and according to (S2) and (S3), $a1 = a(aa) = 1$, i.e., $a \leq 1$ which means that 1 is the greatest element of $(A, \leq)$. According to (S4), $\vee$ is commutative. Because of (S1), (S2) and (S5)

$$a(a \vee b) = a((ab)b) = 1(a((ab)b)) = (((ab)b)((ab)b))(a((ab)b)) = 1,$$

i.e., $a \leq a \vee b$. Hence $b \leq b \vee a = a \vee b$. Because of (S1) and (S5), $a \leq b$ implies

$$(bc)(ac) = ((1b)c)(ac) = (((ab)b)c)(ac) = 1,$$

i.e., $bc \leq ac$. Therefore $a, b \leq c$ implies $cb \leq ab$ and hence

$$a \vee b = (ab)b \leq (cb)b = (bc)c = 1c = c$$

according to (S4) and (S1). Hence $a \vee b$ is the supremum of a and b with respect to $\leq$. If $b \in [a, 1]$ then $ab^a = a(ba) = 1$ according to (S3), i.e., $b^a \in [a, 1]$. Hence ^{a} is a unary operation on $[a, 1]$. If $a \leq b \leq c$ then $c^a = ca \leq ba = b^a$, i.e., ^{a} is antitone. If $a \leq b$ then $(b^a)^a = (ba)a = (ab)b = 1b = b$ according to (S4) and (S1) and hence ^{a} is an involution. Therefore $(A, \vee, (^a; a \in A), 1)$ is an algebra such that $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in A$ $([x, 1], \leq, ^x)$ is a poset with an antitone involution with respect to the partial order induced by $\vee$. Moreover, $((ab)b)b = ab \vee b = ab$ since $b \leq ab$ according to $b(ab) = 1$ by (S3).

Conversely, assume $(A, \vee, (^a; a \in A), 1)$ to be an algebra such that $(A, \vee, 1)$ is a join-semilattice with greatest element 1 and for every $x \in A$ $([x, 1], \leq, ^x)$ is a poset with an antitone involution. Moreover, for all $x, y \in A$ define $xy := (x \vee y)^y$.

$$1a = (1 \vee a)^a = 1^a = a, \tag{S1}$$

$$aa = (a \vee a)^a = a^a = 1, \tag{S2}$$

$$a(ba) = (a \vee (b \vee a)^a)^{(b \vee a)^a} = ((b \vee a)^a)^{(b \vee a)^a} = 1, \tag{S3}$$

$$\begin{aligned}(ab)b &= ((a \vee b)^b \vee b)^b = ((a \vee b)^b)^b = a \vee b = b \vee a \\ &= ((b \vee a)^a)^a = ((b \vee a)^a \vee a)^a = (ba)a, \end{aligned} \tag{S4}$$

$$\begin{aligned}(((ab)b)c)(ac) &= ((((a\vee b)^b \vee b)^b \vee c)^c \vee (a\vee c)^c)^{(a\vee c)^c}\\ &= ((((a\vee b)^b)^b \vee c)^c \vee (a\vee c)^c)^{(a\vee c)^c}\\ &= ((a\vee b\vee c)^c \vee (a\vee c)^c)^{(a\vee c)^c}\\ &= ((a\vee c)^c)^{(a\vee c)^c} = 1. \qquad \text{(S5)}\end{aligned}$$

Therefore $(A,\cdot,1)$ is a strong I-algebra. Moreover, $((a\vee b)^b \vee b)^b = ((a\vee b)^b)^b = a\vee b$ and if $a\leq b$ then $(b\vee a)^a = b^a$. □

Next we define a generalization of the notion of a strong I-algebra:

Definition 2.5. Let $\mathcal{A} = (A,\cdot,1)$ be an algebra of type $(2,0)$. $\mathcal{A}$ is called a *weak I-algebra* if it satisfies

$$1x = x, \qquad \text{(W1)}$$

$$xx = 1, \qquad \text{(W2)}$$

$$x(yx) = 1, \qquad \text{(W3)}$$

$$(xy)y = (yx)x \qquad \text{(W4)}$$

and

$$((xy)y)z = 1 \quad \textit{implies} \quad xz = 1. \qquad \text{(W5)}$$

Remark 2.6. (S5) and (S1) imply (W5).

Theorem 2.7. *Within the definition of a weak I-algebra* (W3) *and* (W5) *may be replaced by the laws*

$$((xy)y)y = xy \qquad \text{(W3}'\text{)}$$

and

$$x((((xy)y)z)z) = 1, \qquad \text{(W5}'\text{)}$$

respectively, and hence weak I-algebras form a variety.

Proof. (W4), (W3) and (W1) imply (W3′):

$$((xy)y)y = (y(xy))(xy) = 1(xy) = xy.$$

(W2) and (W5) imply (W5′):

$$((((xy)y)z)z)((((xy)y)z)z) = 1$$

and hence

$$((xy)y)((((xy)y)z)z) = 1$$

whence

$$x((((xy)y)z)z) = 1.$$

(W3′), (W4) and (W2) imply (W3):

$$x(yx) = ((x(yx))(yx))(yx) = (((yx)x)x)(yx) = (yx)(yx) = 1.$$

Finally, (W1) and (W5′) imply (W5):

$$((xy)y)z = 1 \qquad \text{implies} \qquad xz = x(1z) = x((((xy)y)z)z) = 1. \quad \square$$

Remark 2.8. The variety of weak I-algebras was characterized by the axioms (W1), (W2), (W3′), (W4) and (W5′) in [3] where weak I-algebras were called d-implication algebras.

For the next theorem we need some definitions.

Definition 2.9. An algebra $(A, \sqcup)$ of type (2) is called a *directoid* (cf. [10]) if there exists a partial order relation $\leq$ on A such that for all $a, b \in A$, $a \sqcup b$ is an upper bound of a and b that coincides with $\max(a, b)$ if a and b are comparable. $\leq$ is uniquely determined by $\sqcup$ by $x \leq y$ if and only if $x \sqcup y = y$ $(x, y \in A)$. $(A, \sqcup)$ is called *commutative* if $\sqcup$ is commutative. Let $(P, \leq)$ be a poset with smallest element 0 and greatest element 1 and $f\colon P \to P$. f is called a *switching involution* of $(P, \leq)$ if $f(0) = 1, f(1) = 0$ and $f(f(x)) = x$ or all $x \in P$.

Now we can prove

Theorem 2.10. *Let $\mathcal{A} = (A, \cdot, 1)$ be a weak I-algebra. Define*

$$x \sqcup y := (xy)y \qquad \textit{and} \qquad x^y := xy$$

for all $x, y \in A$. Then $\mathbf{S}(\mathcal{A}) := (A, \sqcup, ({}^x; x \in A), 1)$ is an algebra such that $(A, \sqcup, 1)$ is a commutative directoid with greatest element 1 and for every $x \in A$ $([x, 1], \leq, {}^x)$ is a poset with a switching involution where $\leq$ denotes the partial order induced by $\sqcup$. Conversely, let $\mathcal{S} := (S, \sqcup, ({}^x; x \in S), 1)$ be an algebra such that $(S, \sqcup, 1)$ is a commutative directoid with greatest element 1 and for every $x \in S$ $([x, 1], \leq, {}^x)$ is a poset with a switching involution with respect to the partial order induced by $\sqcup$. Define

$$xy := (x \sqcup y)^y$$

for all $x, y \in S$. Then $\mathbf{A}(\mathcal{S}) := (S, \cdot, 1)$ is a weak I-algebra. Moreover, $\mathbf{A}(\mathbf{S}(\mathcal{A})) = \mathcal{A}$ and $\mathbf{S}(\mathbf{A}(\mathcal{S})) = \mathcal{S}$ for every weak I-algebra $\mathcal{A}$ and every algebra $\mathcal{S} = (S, \sqcup, ({}^x; x \in S), 1)$ such that $(S, \sqcup, 1)$ is a commutative directoid with greatest element 1 and for every $x \in S$ $([x, 1], \leq, {}^x)$ is a poset with a switching involution.

Proof. First assume $\mathcal{A} = (A, \cdot, 1)$ to be a weak *I*-algebra and for all $x, y \in A$ define $x \leq y$ if $xy = 1$, $x \sqcup y := (xy)y$ and $x^y := xy$. Reflexivity and antisymmetry of $\leq$ follow as in the proof of Theorem 2.4. If $a \leq b \leq c$ then $((ab)b)c = (1b)c = bc = 1$ according to (W1) whence $ac = 1$ by (W5), i.e., $a \leq c$. This shows transitivity of $\leq$. Hence $(A, \leq)$ is a poset. That 1 is the greatest element of $(A, \leq)$ and $\sqcup$ is commutative follows as in the proof of Theorem 2.4. Because of (W2) we have $((ab)b)((ab)b) = 1$ whence by (W5) it follows $a((ab)b) = 1$, i.e., $a \leq a \sqcup b$. Hence $b \leq b \sqcup a = a \sqcup b$. If $a \leq b$ then $a \sqcup b = (ab)b = 1b = b$ according to (W1). This shows that $(A, \sqcup, 1)$ is a commutative directoid with 1. That for all $a \in A$, ^{a} is an involution of $[a, 1]$ follows as in the proof of Theorem 2.4. Finally, $a^a = aa = 1$ according to (W2) and $1^a = 1a = a$ according to (W1) showing that ^{a} is switching. Therefore $\mathbf{S}(\mathcal{A}) = (A, \sqcup, (^a; a \in A), 1)$ is an algebra such that $(S, \sqcup, 1)$ is a commutative directoid with greatest element 1 and for every $x \in S$ $([x, 1], \leq, ^x)$ is a poset with a switching involution. Moreover, $((ab)b)b = ab$ follows as in the proof of Theorem 2.4 showing that $\mathbf{A}(\mathbf{S}(\mathcal{A})) = \mathcal{A}$.

Conversely, assume $\mathcal{S} = (A, \sqcup, (^a; a \in A), 1)$ to be an algebra such that $(S, \sqcup, 1)$ is a commutative directoid with greatest element 1 and for every $x \in S$ $([x, 1], \leq, ^x)$ is a poset with a switching involution. Moreover, for all $x, y \in A$ define $xy := (x \sqcup y)^y$.

(W1)–(W4) follow as in the proof of Theorem 2.4.

(W5) If $((ab)b)c = 1$ then

$$
\begin{aligned}
a \leq a \sqcup b \leq (a \sqcup b) \sqcup c &= ((((a \sqcup b)^b)^b \sqcup c)^c)^c \\
&= ((((a \sqcup b)^b \sqcup b)^b \sqcup c)^c)^c = (((ab)b)c)^c = 1^c = c,
\end{aligned}
$$

which implies $ac = (a \sqcup c)^c = c^c = 1$. Therefore $\mathbf{A}(\mathcal{S}) = (A, \cdot, 1)$ is a weak *I*-algebra. $((a \sqcup b)^b \sqcup b)^b = a \sqcup b$ follows as in the proof of Theorem 2.4 and, moreover, $a \leq b$ implies $(b \sqcup a)^a = b^a$ showing that $\mathbf{S}(\mathbf{A}(\mathcal{S})) = \mathcal{S}$. □

Remark 2.11. In [3] commutative directoids with greatest element 1 such that for every $x \in S$ $([x, 1], \leq, ^x)$ is a poset with a switching involution were called *commutative directoids with sectional antitone involutions.*

3. Congruence Kernels

The aim of this section it to characterize congruence kernels of weak *I*-algebras having certain additional properties. First we observe that weak *I*-algebras are *weakly regular* which means that congruences are determined by the class of 1:

Lemma 3.1. *Let* $\mathcal{A} = (A, \cdot, 1)$ *be a weak I-algebra and* $\Theta \in \operatorname{Con} \mathcal{A}$. *Then* $\Theta = \{(x, y) \in A^2 \mid xy, yx \in [1]\Theta\}$.

Proof. If for $a, b \in A$ it holds $a \; \Theta \; b$ then $ab \; \Theta \; aa = 1$ and $ba \; \Theta \; aa = 1$ and if, conversely, $ab \; \Theta \; 1$ and $ba \; \Theta \; 1$ then $a = 1a \; \Theta \; (ba)a = (ab)b \; \Theta \; 1b = b$. □

Next we define the notion of a congruence kernel of a weak I-algebra:

Definition 3.2. A subset K of the base set A of a weak I-algebra $\mathcal{A}$ is called a *congruence kernel* of $\mathcal{A}$ if there exists a congruence $\Theta \in \operatorname{Con} \mathcal{A}$ with $[1]\Theta = K$. Let $\operatorname{Ker} \mathcal{A}$ denote the set of all congruence kernels of $\mathcal{A}$.

Theorem 3.3. *The mappings* $\Theta \mapsto [1]\Theta$ *and* $K \mapsto \{(x, y) \in A^2 \mid xy, yx \in K\}$ *are mutually inverse isomorphisms between* $(\operatorname{Con} \mathcal{A}, \subseteq)$ *and* $(\operatorname{Ker} \mathcal{A}, \subseteq)$ *and hence the latter is a complete lattice.*

Proof. The proof follows immediately from Lemma 3.1. □

Certain subsets of weak I-algebras have a nice property which will be used in the proof of the final theorem of this section. In the following for a subset K of a weak I-algebra $(A, \cdot, 1)$ and an element a of A define $Ka := \{ka \mid k \in K\}$. More generally, for subsets K, L of A we define $KL := \{kl \mid k \in K, l \in L\}$.

Lemma 3.4. *Let* $\mathcal{A} = (A, \cdot, 1)$ *be a weak I-algebra*, K *a subset of* A, $a \in K$ *and* $b \in A$ *and assume* $ab \in K$ *and* $(K(Kx))x \subseteq K$ *for all* $x \in A$. *Then* $b \in K$.

Proof. $b = 1b = (a((ba)a))b = (a((ab)b))b \in (K(Kb))b \subseteq K$. □

Now we can state and prove the characterization of congruence kernels of certain weak I-algebras:

Theorem 3.5. *Let* $\mathcal{A} = (A, \cdot, 1)$ *be a weak I-algebra satisfying*

$$x(yz) = (xy)(xz) \qquad \textit{and} \qquad (xy)((yz)(xz)) = 1$$

and let K *be a subset of* A. *Then* K *is a congruence kernel of* $\mathcal{A}$ *if and only if* $1 \in K$, $AK \subseteq K$ *and* $(K(Kx))x \subseteq K$ *for all* $x \in A$.

Proof. Let $a, b, c \in A$ and $d, e \in K$.

First assume $K \in \operatorname{Ker} \mathcal{A}$. Then there exists a $\Theta \in \operatorname{Con} \mathcal{A}$ with $[1]\Theta = K$ and hence

$$1 \in [1]\Theta = K,$$
$$ad \in [a1]\Theta = [a(aa)]\Theta = [1]\Theta = K$$

and

$$(d(ea))a \in [(1(1a))a]\Theta = [(1a)a]\Theta = [aa]\Theta = [1]\Theta = K$$

proving $1 \in K, AK \subseteq K$ and $(K(Kx))x \subseteq K$ for all $x \in A$.

Conversely, assume $1\in K$, $AK\subseteq K$ and $(K(Kx))x\subseteq K$ for all $x\in A$. Put $\Phi:=\{(x,y)\in A^2\mid xy,yx\in K\}$.

Since $1\in K$, Φ is reflexive.

Obviously, Φ is symmetric.

If $a\ \Phi\ b\ \Phi\ c$ then $ab,ba,bc,cb\in K$ and hence $a(bc)\in AK\subseteq K$ and $(a(bc))(ac)=((ab)(ac))(ac)=(1((ab)(ac)))(ac)\in(K(K(ac)))(ac)\subseteq K$ whence $ac\in K$ according to Lemma 3.4. Interchanging the roles of a and c yields $ca\in K$ and hence $a\ \Phi\ c$. This shows transitivity of Φ.

If $a\ \Phi\ b$ then $ab,ba\in K$ and hence $(ab)((bc)(ac))=1\in K$ whence $(bc)(ac)\in K$ according to Lemma 3.4. Interchanging the roles of a and b yields $(ac)(bc)\in K$ and hence $ac\ \Phi\ bc$. This shows that Φ is a right congruence on $\mathcal{A}$.

If, finally, $a\ \Phi\ b$ then $ab,ba\in K$ and hence $(ca)(cb)=c(ab)\in AK\subseteq K$. Interchanging the roles of a and b yields $(cb)(ca)\in K$ and hence $ca\ \Phi\ cb$. This shows that Φ is a left congruence on $\mathcal{A}$.

Altogether we have proved $\Phi\in\operatorname{Con}\mathcal{A}$. Since, obviously, $[1]\Phi=K$, the proof of the theorem is complete. □

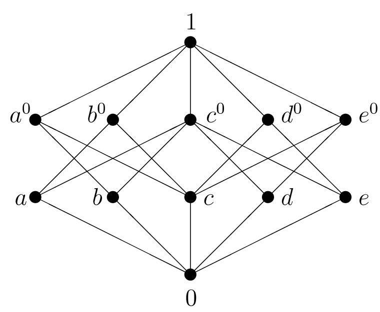
4. Varieties of Implication Algebras

In this section we prove that the different varieties of implication algebras mentioned within the paper form a strictly increasing chain with respect to inclusion.

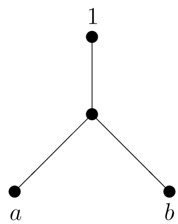
Definition 4.1. Let V_1, V_2, V_3, V_4 and V_5 denote the variety of all implication algebras, orthoimplication algebras, orthomodular implication algebras, strong *I*-algebras and weak *I*-algebras, respectively.

Theorem 4.2. $V_1\subset V_2\subset V_3\subset V_4\subset V_5$.

Proof. The orthoimplication algebra corresponding to $\mathcal{MO}_2:=\mathbf{2}^2+\mathbf{2}^2$ belongs to $V_2\setminus V_1$ since $[0,1]$ is not a Boolean algebra. (Here $+$ denotes the horizontal sum.) The orthomodular implication algebra corresponding to $\mathcal{MO}_2\times\mathbf{2}^1$ with the Hasse diagram



and with $(a^0)^c := d^0$ and $(b^0)^c := e^0$ belongs to $V_3 \backslash V_2$ since $(a^0)^c = d^0 \neq b^0 = a \vee c = (a^0)^0 \vee c$. The strong I-algebra corresponding to the three-element chain belongs to $V_4 \backslash V_3$ since $[0, 1]$ is not an orthomodular lattice. The weak I-algebra corresponding to the poset with the Hasse diagram



with $a \sqcup b := 1$ belongs to $V_5 \backslash V_4$ since it is not a join-semilattice. $\square$

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