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An Extremum Problem for Convex Polygons

By

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Abstract

In the Euclidean plane, consider a convex n-gon with unit perimeter. For a certain class of functions f: $[0, 1/2] \to \mathbb{R}_0^+$ we establish the least upper bound on the sum of the values of f over the distances of all pairs of vertices of the polygon.

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Let f: $[0, 1/2] \to \mathbb{R}_0^+$ be a function. For $n \geq 3$, let $x_1, \ldots, x_n$ be the (pairwise different) vertices of a convex polygon with unit perimeter in the Euclidean plane. Define

$$S_n(f) := \sum_{1 \leq i < j \leq n} f(\|x_i - x_j\|),$$

where $\|\cdot\|$ denotes the Euclidean norm.

We ask, what is the least upper bound on $S_n(f)$?

For the specific function $f(x) = x$ this question was stated as *open problem* in [1] and completely solved in [2, Theorem 1] (here even a best possible lower bound was given). Furthermore, in [2, Theorem 3] an upper bound was given (which is best possible for even n, but not for odd n) if $f(x) = x^2$.

In this short note we give the solution to this question for a certain class of functions. Our result generalizes [2, Theorem 1] and gives the answer to [2, Open Problem 2].

Theorem 1. *Let $f\colon [0,1/2] \to \mathbb{R}_0^+$ be such that the function $x \mapsto f(x)/x$ attains its maximum in $x = 1/2$. Then for any $n \geq 3$ and any convex polygon with n vertices and with unit perimeter in the Euclidean plane we have*

$$S_n(f) \leq f\left(\frac{1}{2}\right)\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil.$$

Furthermore the bound is approached arbitrarily closely by a convex polygon with vertices $x_1,\dots,x_{\lfloor n/2\rfloor}$ that are arbitrarily close to the origin and $x_{\lfloor n/2\rfloor+1},\dots,x_n$ that are arbitrarily close to the point $1/2$ on the x-axis. If the function $x \mapsto f(x)/x$ attains its maximum if and only if $x = 1/2$ and if $n \geq 4$, then the above inequality is even strict.

Remark 1. Note that it is not enough that only f attains its maximum in $x = 1/2$. For example consider the function $f(x) = \sqrt{x}$. If $n = 6m$ points $x_1,\dots,x_{6m}$ are distributed evenly among the vertices of a regular triangle of edge-length $1/3$, then we have $S_{6m}(\sqrt{\cdot}) = \sum_{1\leq i<j\leq 6m}\sqrt{\|x_i - x_j\|} = 12m^2/\sqrt{3} > 9m^2/\sqrt{2} = \sqrt{1/2}\lfloor 6m/2\rfloor\lceil 6m/2\rceil$, such that the bound from Theorem 1 is not valid any more.

For the proof of Theorem 1 we need the following elementary lemma.

Lemma 1. *Let $f\colon [0,1/2] \to \mathbb{R}_0^+$ be such that the function $x \mapsto f(x)/x$ attains its maximum in $x = 1/2$. Let $n \geq 3$ and let $a_1,\dots,a_n$ be the side-lengths of a plane convex n-gon with perimeter at most one*, i.e., *$\sum_{k=1}^n a_n \leq 1$. Then we have*

$$\sum_{k=1}^n f(a_k) \leq 2f\left(\frac{1}{2}\right).$$

Proof. Trivially we have $a_k \leq 1/2$ for all $k = 1,\dots,n$ and therefore

$$\sum_{k=1}^n f(a_k) = \sum_{k=1}^n \frac{f(a_k)}{a_k} a_k \leq 2f\left(\frac{1}{2}\right)\sum_{k=1}^n a_k \leq 2f\left(\frac{1}{2}\right). \qquad \square$$

Now we give the proof of Theorem 1.

Proof of Theorem 1. We use the ideas from [2, Proof of Theorem 1]. Let the vertices $x_1,\dots,x_n$ of the polygon P be arranged clockwise.

Assume first that n is even. Now we consider the $\binom{n/2}{2}$ convex quadrangles

$$Q_{i,j} := \overline{x_i x_j x_{i+\frac{n}{2}} x_{j+\frac{n}{2}} x_i}$$

for all i and j satisfying $1 \leq i<j \leq \frac{n}{2}$. Let

$$u(i,j) := f(\|x_i - x_j\|) + f(\|x_j - x_{i+\frac{n}{2}}\|) + f(\|x_{i+\frac{n}{2}} - x_{j+\frac{n}{2}}\|) \\ + f(\|x_{j+\frac{n}{2}} - x_i\|).$$

As $Q_{i,j}$ is convex and contained in P it follows from Lemma 1 that $u(i,j) \leq 2f(1/2)$. Trivially $\|x_i - x_{i+n/2}\| \leq \frac{1}{2}$ for all i such that $1 \leq i \leq \frac{n}{2}$ and $\|x_i - x_{i+n/2}\| < \frac{1}{2}$ for at least one such choice of i. Since also f attains its maximum in $x = 1/2$ we have

$$S_n(f) = \sum_{1\leq i<j\leq\frac{n}{2}} u(i,j) + \sum_{i=1}^{\frac{n}{2}} f(\|x_i - x_{i+\frac{n}{2}}\|) \leq \binom{\frac{n}{2}}{2} 2f\left(\frac{1}{2}\right) \\ + \frac{n}{2} f\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right)\frac{n^2}{4}.$$

Here the first equality can be easily checked by counting all the different distances occurring on the right-hand side. It is clear that each distance on the right side of the equality appears at most once, but on the other hand we sum up $4\binom{n/2}{2} + \frac{n}{2} = \binom{n}{2}$ distances, so the equality is true. Furthermore, if $f(x)/x < 2f(1/2)$ for all $x \in [0, 1/2)$, then also $f(x) < f(1/2)$ for all $x \in [0, 1/2)$ and the above inequality is strict as well.

Now let n be odd. In this case we consider the $\binom{\frac{n-1}{2}}{2}$ convex quadrangles

$$Q_{i,j} := \overline{x_i x_j x_{i+\frac{n+1}{2}} x_{j+\frac{n+1}{2}} x_i}$$

for $1 \leq i<j \leq \frac{n-1}{2}$ and the $\frac{n-1}{2}$ convex triangles

$$R_i := \overline{x_i x_{\frac{n+1}{2}} x_{i+\frac{n+1}{2}} x_i}$$

for $1 \leq i \leq \frac{n-1}{2}$. Let

$$u(i,j) := f(\|x_i - x_j\|) + f(\|x_j - x_{i+\frac{n+1}{2}}\|) + f(\|x_{i+\frac{n+1}{2}} - x_{j+\frac{n+1}{2}}\|) \\ + f(\|x_{j+\frac{n+1}{2}} - x_i\|)$$

and

$$v(i) := f(\|x_i - x_{\frac{n+1}{2}}\|) + f(\|x_{\frac{n+1}{2}} - x_{i+\frac{n+1}{2}}\|) + f(\|x_{i+\frac{n+1}{2}} - x_i\|).$$

As $Q_{i,j}$ and R_i are both convex polygons and contained in P, it follows from Lemma 1 that $u(i,j) \leq 2f(1/2)$ and $v(i) \leq 2f(1/2)$. So

$$S_n(f) = \sum_{1 \leq i < j \leq \frac{n-1}{2}} u(i,j) + \sum_{i=1}^{\frac{n-1}{2}} v(i) \leq 2f\left(\frac{1}{2}\right)\left(\binom{\frac{n-1}{2}}{2} + \frac{n-1}{2}\right)$$
$$= f\left(\frac{1}{2}\right)\frac{n^2-1}{4}.$$

Here the first equality can be checked as above. If $n > 3$, then we must have $v(i) < 2f(1/2)$ for at least one i satisfying $1 \leq i \leq \frac{n-1}{2}$.

Hence the desired bound is proved in both cases.

It is easy to see that the bound is approached arbitrarily closely by the point distributions given in Theorem 1. We just mention that $\lim_{x \to 0^+} f(x) = 0$ since $0 < f(x)/x \leq 2f(1/2)$ for all $x \in (0, 1/2]$. $\square$

References

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